
Learning Math with Graphing Calculator

Investigating Symbols, Functions, and Graphs

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Graphing Calculator

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Learning Math with Graphing Calculator

Tim Erickson

Table of Contents

Introduction.....	1
Which version do you have?.....	1
How do you learn?	2
The Basics of Graphing Calculator.....	3
Anatomy of the Graphing Calculator Window.....	3
How do I zoom the graph in or out?.....	3
How do I see other parts of the graph?.....	4
How do I save my work?	4
What if I need help?	4
Using the Math Window.....	7
How do I enter an equation?.....	7
How do I select?	7
What do I need to know about entering formulas?.....	9
Are there other ways to enter math stuff?.....	10
How do I do fractions?	11
How do I make exponents?.....	11
Can I use Graphing Calculator as a calculator?	12
What about simple algebra?	13
Can I multiply out two parenthetical expressions?.....	16
What about plain old arithmetic?	16
How do I get more (or fewer) significant figures?	17
How do I isolate a term?.....	17
Can I do trig functions?	18
Graphing	20
How do I make a graph?.....	20
How do I see different values for my parameters?.....	20
Can I graph more than one function?	22
Can I trace functions?.....	22
Do I have to use x and y—and n? Can I use other letters?.....	22

What should I know about constants and parameters?	23
Graphing in Two Dimensions	24
Polar coordinates.	24
Inequalities	26
Transformations	34
Translations, dilations, and $y = mx + b$	35
Dilations: stretching, squishing, and flipping	37
Putting transformations together.	38
Once more, with waves.	40
Choosing a parameterization	41
Translations, the minus sign, and symmetry.	41
The difference between horizontal and vertical	42
Exponentials	44
Adding functions part I.	45
Adding functions part II	46
Difference functions	47
Fourier series.	48
Multiplying functions: envelopes	50
Reciprocals of functions	52
Inverse functions.	53
Functions of functions: composition.	55
Surfaces in 3D	60
Basic 3D drawing in Graphing Calculator	60
How to tell which axis is which	61
Cool ways to slow down your computer	61
A Family of Graphs at a Glance	63
Surfaces as Genuine Functions of Two Variables	65
Surfaces of revolution.	65
Polar Coordinates and Surfaces	68
Slicing 3D Surfaces	70
Challenges.	72
Conic sections—using cones!	74
More general 3D surfaces	77
Exploring Parametric Equations.	79
Remind me about parametric equations	79
Making parametric curves in Graphing Calculator.	79
Lissajous fun.	81
Parametric equations without trig functions.	82
Making a spring: 3D parametric curves	82

Parametric Surfaces	84
Making the torus	84
Adding lines for understanding	87
Changing the donut into a double helix	88
Fancy Stuff	91
Coloring	91
Conditions	93
If statements	93
Summations	95
Functions	98
Symbolic calculus	99
The difference between total and partial derivatives	101
Ordinary differential equations	102
Integrals	103
Signum: a Useful Function to Know About	103
Vector fields	104
But wait! There's more!	106
Solving Problems with Graphing Calculator	107
Word Problems: the Big Picture	107
What Symbol Manipulation Is For	108
Dimensional Analysis	109
Using Substitution	113
Figuring Average Speed	113
More About Symbol Manipulation	116
Two Extended Examples	120
Amanda's Shadow Investigation	124
Proportion Problems	127
Maximum-Minimum Problems	129
Simultaneous Equations	132
Equilibrium, Titration, and Logarithmic Axes	136
Orbital Mechanics	141

Solutions..... **144**

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Introduction

Graphing Calculator is a graphing program. It graphs functions and relations, implicit and explicit, parametric and “straight,” in two and three dimensions. It knows a wide array of mathematical functions already, including all the trig functions.

You speak to the program in equations and formulas—you communicate symbolically. You can use named parameters in your expressions:

$$y = mx + b$$

instead of just

$$y = 3x - 1$$

You can even make parameters into sliders and drag to change values.

Which version do you have?

The most recent versions of the Graphing Calculator software are at <http://www.PacificT.com/FreeStuff.html>. At the time of this writing, the following versions are the most current:

- ◆ On both the Macintosh and on Windows, Graphing Calculator 3.2 is the fully functional commercial release.
- ◆ Apple bundles a limited version (Graphing Calculator 1.3) with the Macintosh. Most of the material in this book can be worked with this free version. Features available only with the commercial version are indicated.

Pacific Tech also offers a free reader application, Graphing CalculatorViewer for both Macintosh and Windows, so you can publish Graphing Calculator documents on the Web.

If you have Graphing Calculator 1.3 and you try an example that isn't supported in that version, Graphing Calculator will advise you to visit www.PacificT.com to order version 3.

How do you learn?

Some people learn by messing around. Some learn when they have a specific job to do. Some—believe it or not—read manuals.

A lot of information about Graphing Calculator appears on the screen in a message or in on-line help, and you can learn a lot by watching the demos and playing with the examples.

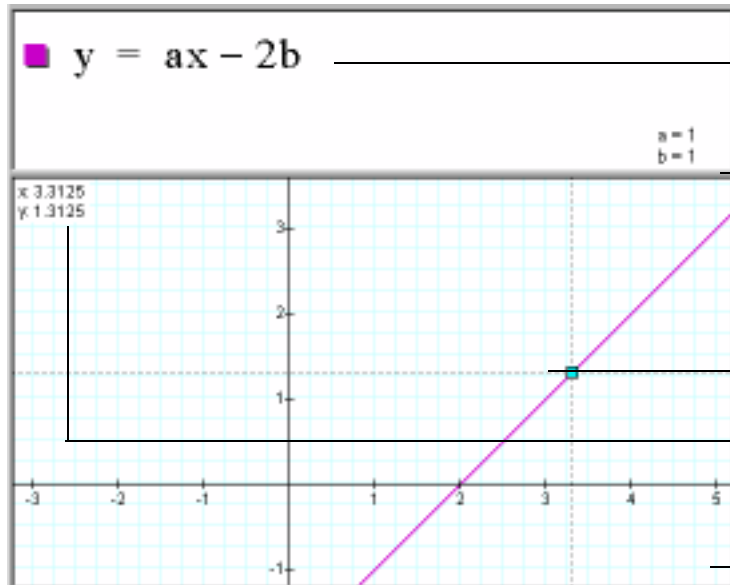
This book lets you benefit from experience. It can show you cool things you may not find in the help, help you avoid—or deal with—pitfalls, and point you to ways of doing things that work well.

Best of all, this book can help you learn about mathematics, using Graphing Calculator as a tool. There are situations where technology makes a qualitative difference in understanding concepts.

The Basics of Graphing Calculator

Anatomy of the Graphing Calculator Window

Welcome to the world of Graphing Calculator. This is what it looks like:



This is the math pane.

This divider separates the math pane from the graph pane. You can drag it up and down.

Click a curve to place a “tracer,” then drag to trace the curve.

Tracer coordinates appear in the upper-left corner of the graph pane.

This is the graph pane.

How do I zoom the graph in or out?

Use the little-mountain/big-mountain buttons at the bottom of the window. Here are some other things you need to know:



- ❖ You can use the zoom buttons to stretch and shrink the horizontal axis independently. On the Macintosh, do this by holding down the **option** key while you click the zoom buttons. On Windows, hold down the **ctrl** key.
- ❖ You can use the zoom buttons to stretch and shrink the vertical axis independently. On the Macintosh, do this by holding down the **⌘** key while you click the zoom buttons. On Windows, hold down the **shift** key.
- ❖ If you have a tracer on a curve, the zooming centers on the trace point. This is extremely useful for investigating special places on a function.

How do I see other parts of the graph?

- ◆ If you grab an axis (not a curve), you can move (pan) the display anywhere you like. If the actual axis is not in the window, you'll see a phantom axis that shows you the coordinates. You can drag it the same way.
- ◆ If you drag your cursor off the edge, the graph will keep panning.
- ◆ You can always choose **Reset Graph** from the **Graph** menu (on the Macintosh the shortcut is **⌘-r**; on Windows it's **ctrl-r**) to reset the axes to their original, close-to-the-origin state.

How do I save my work?

Use **Save** in the **File** menu. You can use **Copy** or **Copy As Text** on the **Edit** menu to copy equations from the math pane and paste them into most word-processing programs. To paste your graph into another application, use **Copy Graph** on the **Edit** menu.

When you paste equations back into Graphing Calculator—as long as you leave them the original size—everything still works. And if you paste the picture in, it remembers the equation. If you're using a Macintosh with System 7.5 or later, you can also drag a graph off of the Graphing Calculator window to create a clipping document. Drag the clipping file back into the math pane to restore it.

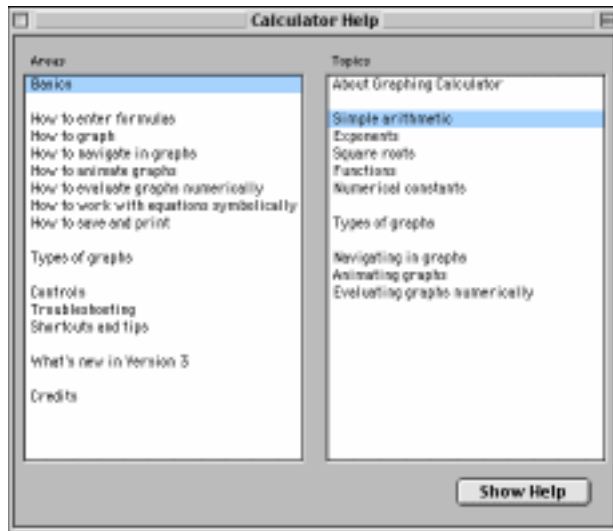
What if I need help?

Graphing Calculator's help system lists all the commands and gives good suggestions of things to do to get acquainted with the program. The help system is a little different on the Macintosh and on Windows.

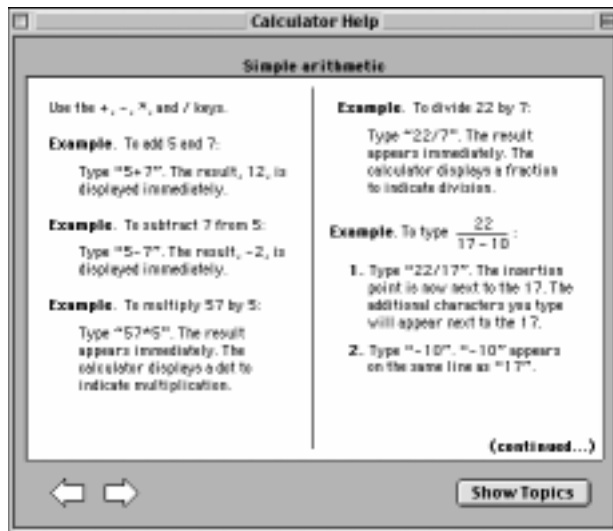
On the Macintosh, there's a list of areas; for each area there's a list of topics.

On either platform, you start by choosing **Graphing Calculator Help** from the **Help** menu.

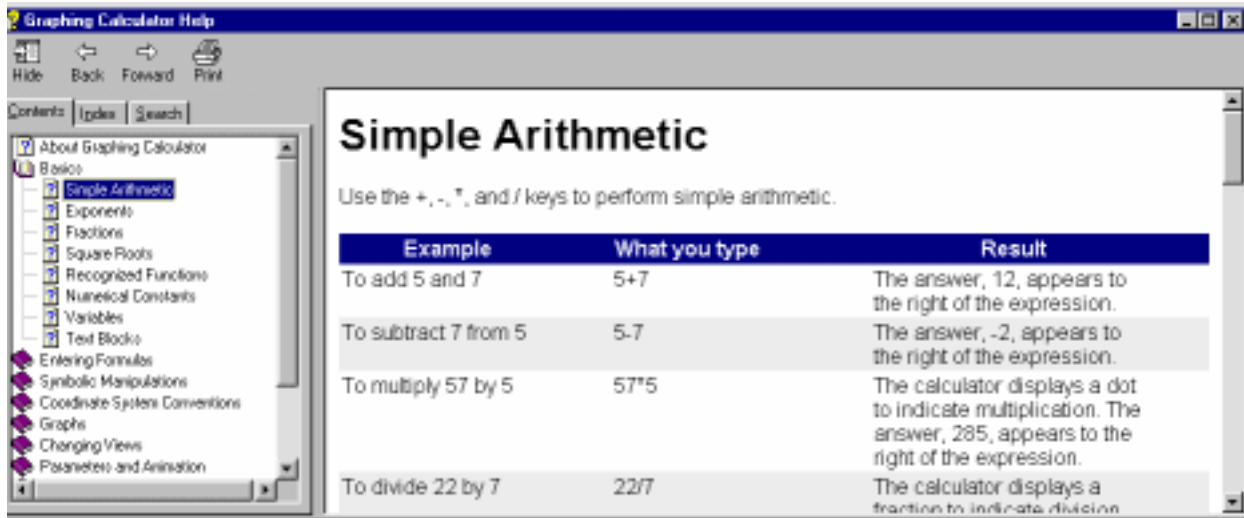
Here is the window that appears on the Macintosh:



To use help, select an area on the left and a topic on the right, then click **Show Help**. The instructions will appear in the window.



On Windows, you'll see this:



Use the table of contents, index, or search tab at the left to display the topic you want on the right.


Using the Math Window

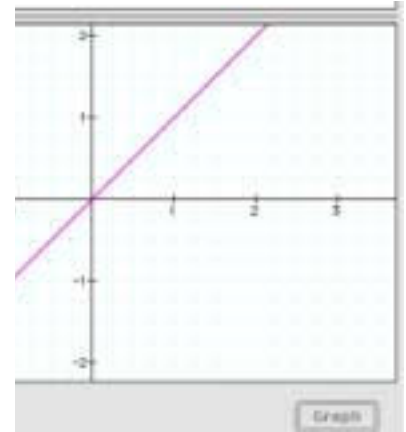
We'll begin a series of “How do I...” questions—the ones teachers tend to ask when they see the program.

How do I enter an equation?

This is a great way to introduce the capabilities of the program. Start by creating and graphing the graph shown at the right.

- ❖ Start Graphing Calculator.
- ❖ Type $y = x$ and press **enter**.

If you have a Macintosh version of Graphing Calculator, you can also draw a graph by pressing the **return** key or clicking the  button. In the Windows version, you can choose **Draw Graph** on the **Graph** menu. Pressing **enter** works in all versions.



How do I select?

Selection is key to entering the formulas into Graphing Calculator. Here are some rules to keep in mind:

- ❖ If you're editing a formula, clicking the mouse moves the *insertion point*, which appears as a flashing vertical bar.
- ❖ Dragging the mouse or shift-clicking selects (highlights) a portion of the formula.
- ❖ You never have an insertion point and a selection at the same time.
- ❖ If there's an insertion point, typing inserts what you type at that point.
- ❖ If there's a selection, it's more complicated. See the description on the next page.

You can select by dragging with the mouse, and you can also use commands and arrow keys. **Select All** on the **Edit** menu selects the entire expression. Right-arrow moves through the expression. Shift-arrow and shift-click “extend” selections. The up-arrow key selects the next larger “chunk” of an expression. For example,


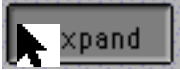
$(3x|+4)^2$ plus the up arrow makes $(3x+4)^2$

$(3x+4)^2$ plus the up arrow makes $(3x+4)^2$

$(3x+4)^2$ plus the up arrow makes $(3x+4)^2$

If something is selected and you type a number or a letter, what you type replaces the selection. But if you type an *operator*, the selection becomes the *operand*.

Operators are things like addition, division, exponentiation, absolute value, square root, and any of the algebra commands. Some examples will make it clear:

If you have	and you type	you get
$y = x + 1$	w	$y = w$
$y = x + 1$	27	$y = 27$
$y = x + 1$	*7	$y = 7(x + 1)$
$y = x + 1$	($y = (x + 1)$
$y = x + 1$	Option-r (on the Macintosh) or ctrl-shift-R (on Windows)	$y = \sqrt{x + 1}$
$y = x + 1$! (factorial)	$y = (x + 1)!$
$y = x + 1$	sin	$y = \sin$
$y = x + 1$	 (on the keypad)	$y = \sin(x + 1)$
$y = (x + 1)^2$	 (on the keypad)	$y = x^2 + 2x + 1$

Try the following steps to enter $(x + a)^2 - 1$ —without using parentheses.

- ❖ Type $x + a$
- ❖ Press **up arrow** twice. Now “x+a” is selected.
- ❖ Type 2 .

Graphing Calculator puts in the parentheses since the exponent operates on the whole selected expression. Notice how the insertion point is raised—it’s up in “exponent land.”

- ❖ Press **right arrow**. The insertion point moves down.
- ❖ Type -1 .
- ❖ Press **enter**.

Graphing Calculator will try to graph your equation. It succeeds, assuming (since you didn’t type a full equation and didn’t specify a) that $a = 1$ and that you meant

$$y = (x + a)^2 - 1.$$

What do I need to know about entering formulas?

Enter formulas from the keyboard. When you press **enter**, Graphing Calculator graphs whatever you typed, if it knows how. Math being what it is, it’s not always obvious how to get what you want.

Addition, subtraction, and multiplication are easy. Use + or - for addition and subtraction. Usually you don’t have to use anything for multiplication (although * works).

Division

Division is trickier, because you have to be careful about what’s above and what’s below the line. Consider this sequence of keystrokes:

x+1/2

Which do you mean:

$$x + \frac{1}{2} \text{ or } \frac{x + 1}{2}$$

Graphing Calculator thinks you mean the first one, not the second. You can get the second in two different ways—using selection (see the previous section) and using parentheses.

Parentheses

Use parentheses whenever you need to avoid ambiguity. To make a pair of parentheses, type (. If an expression is selected, this parenthesizes the selection.

- ❖ Starting with a clean window, type (.
 To create a clean window, choose **New** from the **File** menu.
- ❖ Notice that both parentheses appear.
- ❖ Now type $x+1)$ even though the close parenthesis is already there.
 Typing a close parenthesis pops you out of the parentheses.
- ❖ Finally, type $/2$.

Got the idea? The division operator applies to the preceding factor. Since the factor is in parentheses, the division operator applies to the whole expression.

Are there other ways to enter math stuff?

You can enter functions from the **Math** menu, and you can also use the *keypad*. Some people like to keep it open all the time. It has modifier buttons, just like a fancy calculator, to change the meanings of some keys. The regular keypad is on the left—you get the keys on the right when you press the **Inverse** button.

- ❖ Choose **Show Keypad** from the **Math** menu.
 Square root is missing! No problem.
- ❖ Press the **Inverse** button to change the relevant buttons to their inverses.



Note: This is how you can take arbitrary roots (not just square ones), as well as cool stuff like log base 2. This keypad may also remind you that Graphing Calculator can do absolute value and factorial (which you can also do by pressing the vertical bar key or the exclamation point, respectively).

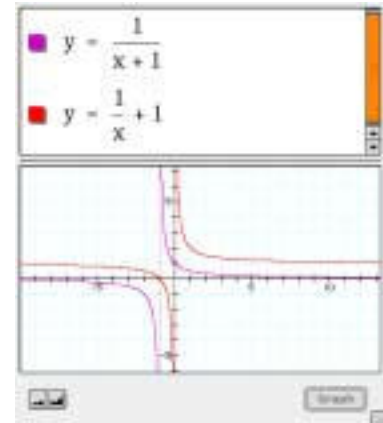
■ $\log_2 32 = 5$
■ $\sqrt[25]{32} = 4$
■ $10! = 3628800$

How do I do fractions?

Here are some principles to follow:

- ❖ Use the slash key (/). Graphing Calculator makes “actual” fractions—the numerator is over the denominator.
- ❖ To exit a fraction, click where you want the cursor to be or press the **right arrow** key.
- ❖ To make a whole expression into a numerator, select the expression before pressing the slash key.
- ❖ When in doubt, use parentheses. See “How do I select?” on page 7.

Try to get the two functions shown in the illustration to appear at once in your Graphing Calculator window.



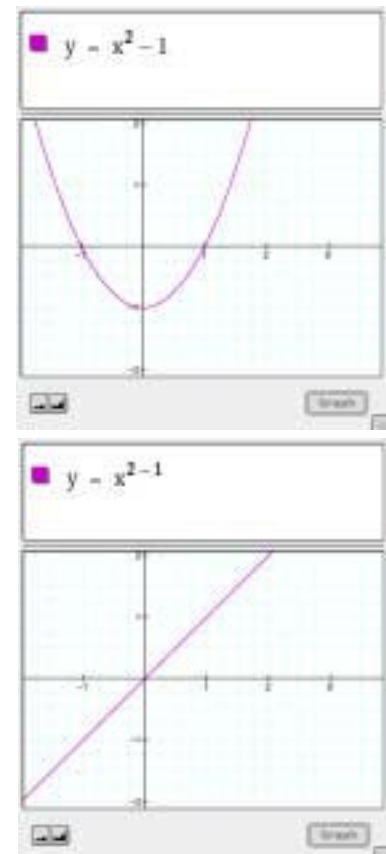
How do I make exponents?

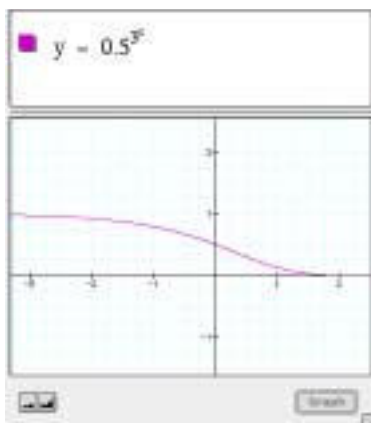
Suppose you want $y = x^2 - 1$.

- ❖ Select everything and press **delete** (or **backspace**) to erase it.
- ❖ Type $y = x \wedge 2$ (The \wedge is **shift-6**, the circumflex or caret.)
- ❖ Press the **down arrow** key and type -1 .
- ❖ Press **enter** to graph the function.

The caret, $\langle \wedge \rangle$, moves you up into “exponent land.” To get out, press **right arrow** \Rightarrow on your keyboard. If you forgot to, you would graph $y = x^{2-1}$. The difference is plain.

You can make exponents of exponents as well:





Can I use Graphing Calculator as a calculator?

If you just type numbers and operations, Graphing Calculator will show you the current value of whatever you've typed. If it's really big or small, Graphing Calculator will use scientific notation. Try it! Start with a fresh screen for each of the following:

- ◆ Type $2+2$.
- ◆ Type $30!$ (thirty factorial).
- ◆ On a Macintosh, hold down \mathbb{R} while pressing **r**, and type **3**. On Windows, hold down **shift** and **ctrl** while pressing **r**, and type **3**.
- ◆ Type 2^{10} .
- ◆ Type $1/0$.

Graphing Calculator also knows some constants.

- ◆ Type **pi**.

Things to do

- ◆ How big is the biggest number (besides infinity) that Graphing Calculator can represent?
- ◆ How small is the smallest *positive* number?
- ◆ Is the range of negative numbers any different?
- ◆ The number of ways you can choose a 5-member basketball squad from among 32 players is $\frac{32!}{(32-5)!5!}$

How many is that?

- ◆ How many seconds have you been alive?

What about simple algebra?

Graphing Calculator can help you solve equations and do simple algebra. You still have to know what you're doing, but Graphing Calculator can help you do the bookkeeping. You need to learn four commands in the **Math** menu (or the keypad) plus one great mouse technique, called *drag algebra*. Let's do drag algebra first.

You can select an expression in the math pane, grab it with the mouse, and drag it left or right. Dragging preserves the value of an expression and the truth of an equation.

What happens depends on the situation. An example will help.

- ◆ Type the Pythagorean identity ($a^2 + b^2 = c^2$) into the math pane.
- ◆ Select the first term. You'll see:

$$a^2 + b^2 = c^2$$

- ◆ Grab it with the mouse pointer, which turns into a fist. Drag the term to the left.

$$a^2 \left(1 + \frac{b^2}{a^2} \right) = c^2$$

- ◆ Now try moving it to the right. You'll see more equations, all the result of moving the a^2 term around:

$$b^2 + a^2 = c^2$$

$$b^2 = -a^2 + c^2$$

$$b^2 = c^2 - a^2$$

$$b^2 = \left(-\frac{c^2}{a^2} + 1 \right) (-a^2)$$

Drag algebra is good for factoring terms out of an expression (as in the first and last cases) or rearranging them.

You don't have to select the whole term. Try selecting b (not b^2) and moving it around. You should be able to produce this:

$$\frac{a^2}{b} + b = c^2 \quad \text{and this:} \quad \frac{(ab^{1/2})^2}{b} + b^2 = c^2$$

among many others.

- ◆ To start over, select (drag over) what's in the math pane (above the double line, where the equations are) and delete it.
- ◆ Type a^2

You should see this:

$$a^2$$

To exit the exponent, press the right arrow.

- ◆ If we just type the plus sign, it will go up in the exponent—press the right arrow, as advised.

Now you should see:

$$a^2|$$

Notice that the cursor is taller now.

- ◆ Type $+b^2$ (**right arrow**) = c^2 .
- You should see the formula we want in the math pane. Okay, now the cool stuff.
- ◆ Select the letter b by dragging the mouse pointer over it.

$$a^2 + b^2 = c^2$$

- ◆ Select **Isolate Term** from the **Math** menu.

Shazam! You should see this: $b = \sqrt{c^2 - a^2}$

How do I get my original equation back?

You can always use **Undo**, but what if you forget? You drag the b where it belongs. You can't get any further. What can you do?

$$a^2 + bh = c^2$$

- ◆ Select what you want to simplify (either bb or something that contains it) and choose **Simplify** from the **Math** menu or from the keypad.

$$a^2 + bb = c^2 \text{ becomes } a^2 + b^2 = c^2$$

In general, you can use **Simplify** to combine like terms, either through addition or multiplication.

Things to do

Use drag algebra to:

- ◆ Turn $b = 3x + 4x$ into $b = x(3 + 4)$.
- ◆ Turn $x + y = 5$ into $y = 5 - x$.
- ◆ Turn $f = 2a + 4b$ into $f = 2(a + 2b)$.
- ◆ Turn $f = 2a + 4b$ into $f = 4\left(\frac{a}{2} + b\right)$.
- ◆ Turn $\frac{1}{1+x} + \frac{x}{1+x}$ into $\frac{1+x}{1+x}$.

Hint: Drag the denominator first.

- ◆ Turn $\frac{1}{x} + 3 + 4$ into $3x + 4x = 1$.

Use drag algebra and **Simplify** to:

- ◆ Turn $2x + 3x$ into $5x$.
- ◆ Turn $\frac{x}{2} + x$ into $3\frac{x}{2}$.
- ◆ Turn $2x + 2y = 5x$ into $2y = 3x$ and then into $y = \frac{3x}{2}$.
- ◆ Turn $x + (x + 1)(x - 1) = 2x + 1$ into $x = 2$.

Hint: Graphing Calculator can't factor trinomials, so watch out! Can you get the other solution, $x = -1$?

◆ Turn $\frac{1}{3} + \frac{1}{4}$ into $\frac{7}{12}$.

This turns out to be easy, but beware: Graphing Calculator doesn't add fractions so easily if there are variables present.

◆ Turn $\frac{a}{3} + \frac{2a}{3}$ into a .

Hint: Use the distributive property.

Can I multiply out two parenthetical expressions?

If for some reason you had the Pythagorean identity written this way:

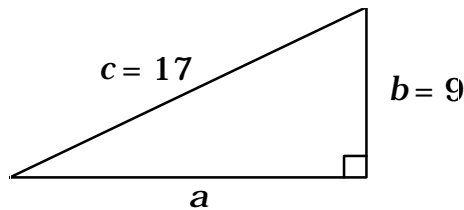
$$b^2 = (c + a)(c - a)$$

and you wanted the right side multiplied out, you could select it and choose **Expand** from the **Math** menu.

$$b^2 = (c + a)(c - a) \text{ becomes } b^2 = c^2 - a^2$$

What about plain old arithmetic?

When you have arithmetic to do, use **Calculate**. For example, suppose you have a specific right triangle in mind:



You would enter $a = \sqrt{17^2 - 9^2}$

What is the value of the right-hand side? Select it (or the whole thing) and choose **Calculate** from the **Math** menu or the keypad.

$$a = \sqrt{17^2 - 9^2} \text{ becomes } a = 14.4222$$

If you want to leave it in “root” form, just select the inside. Then

$$a = \sqrt{17^2 - 9^2} \text{ becomes } a = \sqrt{208}$$

How do I get more (or fewer) significant figures?

Choose **Preferences** from the **Edit** menu. You'll get the box in the margin.

Use it to change the number of significant figures as well as other fun things.

Note: If you change the **3D Background** to **Room** or **Floor**, 3D graphs will plot more slowly.



Things to do

Use Graphing Calculator and its algebra commands to answer the following:

- ❖ Write $(a + b)^4$ as a polynomial.
- ❖ Write $(p + q + r)^2$ as a polynomial.
- ❖ If two sides of a right triangle have lengths 10 and 11, what are the possible lengths of the other side?
- ❖ The Earth’s circumference is 40,000 km. What is its volume?
- ❖ What are the solutions to $x^2 - 3x - 3 = 0$?

Use the quadratic formula. Give both solutions to three decimal places. Then give them in “root” form. Verify with a graph.

- ❖ What is $\sin(1)$? Can you draw a picture of the unit circle that shows where that number appears?

Remember: Graphing Calculator uses radians.

How do I isolate a term?

The **Isolate Term** command is amazing. Select something—anything—and choose **Isolate Term**. It will go to one side of the equals sign, and everything else will be on the other. Three examples from Pythagoras:

$$a^2 + b^2 = c^2 \text{ becomes } a^2 = c^2 - b^2$$

$$a^2 + b^2 = c^2 \text{ becomes } a = \sqrt{c^2 - b^2}$$

$$a^2 + b^2 = c^2 \text{ becomes } 2 = \log_a(c^2 - b^2)$$

However:

$$x^2 + 5x - 6 = 0 \text{ becomes } x = \sqrt{-5x + 6}$$

The result may be true, but might not have been what you hoped.

Things to do

Use Graphing Calculator to solve the first ten equations for x :

$$\diamond y = 2x + 3$$

$$\diamond y = \sqrt{x+2}$$

$$\diamond y = \frac{2}{\sqrt{1+x^2}}$$

$$\diamond y = (x+4)^2 - 8x$$

$$\diamond x + a = \frac{1+x}{a}$$

$$\diamond x^2 - 9 = (x^2y - 3xy)(x+3)$$

$$\diamond y = \tan x$$

$$\diamond y = e^{-x}$$

$$\diamond y = e^{\sqrt{x}}$$

$$\diamond \frac{1}{x-1} = \frac{2}{3x-5}$$

Use **Help** to figure out what **Transform** does.

Can I do trig functions?

Just type the name of the function and an argument to enter it. Here are some things you may want to know:

- ❖ If your argument is just x or a single term in x , you don't need parentheses, but they're useful if you add another term.
- ❖ To square one of these functions, square the whole thing—don't try to enter the traditional notation $y = \sin^2 x$. Type $y = (\sin x)^2$, and Graphing Calculator will reformat it into the right notation. (See “How do I select?” on page 7.)
- ❖ Don't use a negative exponent to indicate the inverse function—you'll get a reciprocal. Instead, use the name of the inverse. That is, for the inverse of the sine, use $y = \text{asin } x$, not $y = \sin^{-1} x$.
- ❖ Trig functions are in radians, so you may need π . Type “pi” to get the Greek letter—same for theta and phi.

Trigonometric	sin, cos, tan, sec, csc, cot	
Hyperbolic	sinh, cosh, tanh, sech, csch, coth	
Inverses of above	asin, asinh, etc	
Logarithms	log, ln	log base 10 and natural log, respectively
Exponential	exp	(“e-to-the”-the inverse of ln)
Others	sgn	signum: -1 if x is negative, +1 if it's positive, 0 if 0.
	floor	if not an integer, the next lowest integer.
	ceil	if not an integer, the next highest integer

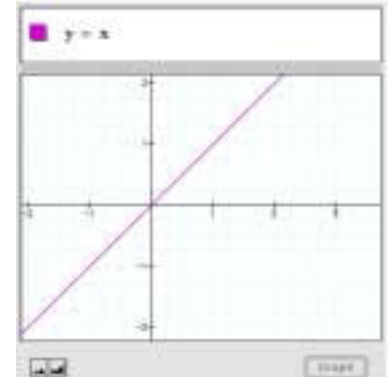
Graphing

We will not explain graphs in this booklet, but if you're an algebra student or teacher, or if you ever were one, you probably remember something about x and y axes. The idea was to represent an equation (such as $y = x$) as a graph (in this case, a straight line). It's absurdly easy in Graphing Calculator.

How do I make a graph?

Here are the step-by-step instructions to create a simple graph:

- ❖ Start the program
- ❖ Type $y=x$
- ❖ Press the **enter** key, choose **Draw Graph** from the **Graph** menu, or press the **Graph** button.



Note: There is no **Graph** button in the Windows version of Graphing Calculator, and there is no **Draw Graph** command on the **Graph** menu for the Macintosh. However, pressing **enter** always works, and on the Macintosh you can also press the **return** key.

The graph will be a slanty purple line like the one in the top picture on this page.

How do I see different values for my parameters?

Graphing Calculator is a *dynamic* graphing tool. It supports building understanding through seeing how things change. So try this:

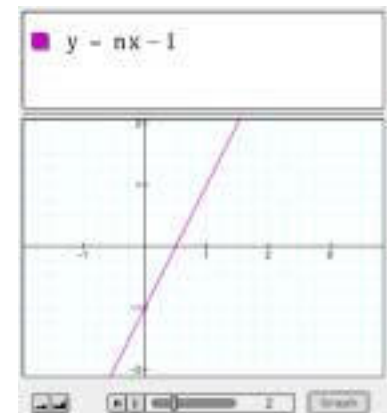
Edit the equation to read $y = nx - 1$

- ❖ Press the **backspace** key once.
- ❖ Type **$nx-1$** .
- ❖ Press **enter**.

Using the Animation Slider

A slider appears at the bottom, showing the value for n .

- ❖ Drag the “knob” on the slider. See how n changes—and how the graph changes as well.

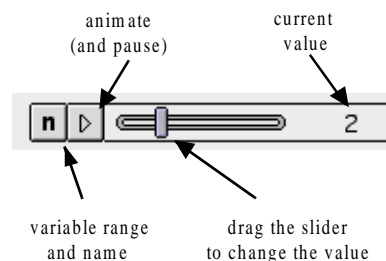


- ◆ Press the animation button (the triangle) at the left of the slider to make n change automatically.
- ◆ Press the pause button to make it stop.

The animation cycles through a series of values. You can change those values.

- ◆ Stop the animation by pressing the pause button.
- ◆ Press the “n” button.

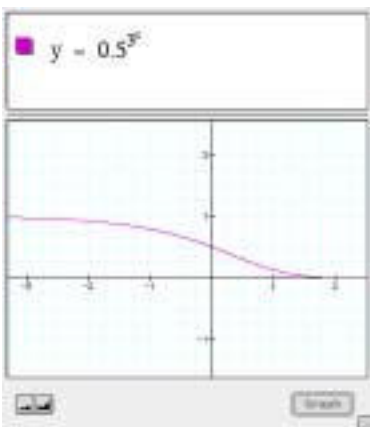
You’ll see a dialog where you can control which values n gets—and even which letter (besides n) appears on the slider.



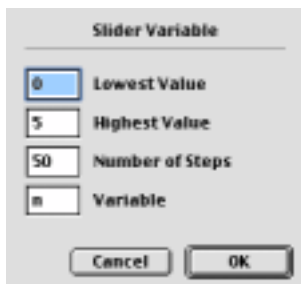
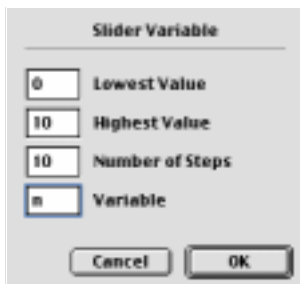
Things to do

- ◆ Change the numbers for the **Lowest Value**, **Highest Value**, and **Number of Steps**.
- ◆ Try making the **Lowest Value** negative.

Predict first, then see if you are right.



- ◆ Press the “n” button. The dialog on the left is the default, but you can change the numbers. Try the ones on the right.



- ❖ Suppose you wanted to explore lines with different slopes. Try $y = nx$. Try out different limits in the box. Can you make the line spin all the way around?

Can I graph more than one function?

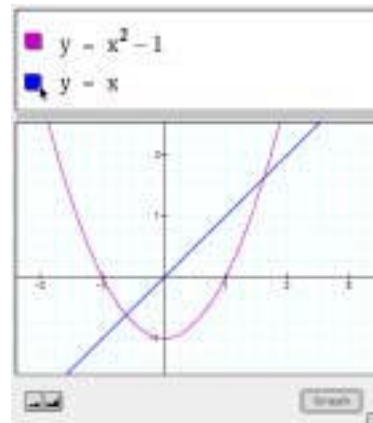
Sure. Let's graph two functions together:

- ❖ **Graph** $y = x^2 - 1$ as before.
- ❖ Choose **New Math Expression** from the **Math** menu.

A new “block” appears. (The shortcut is \mathbb{H} -m on the Macintosh or **ctrl-m** on Windows.)

- ❖ Type $y = x$ and press **enter**. Now you see both functions.

When you click the block, a pop-up menu of colors appears and you can choose a color for your function. We chose blue for the line.



Things to do

- ❖ Enter the following expressions, and graph them if you can:

1. e^{-x}

2. $F = \frac{GMm}{r^2}$

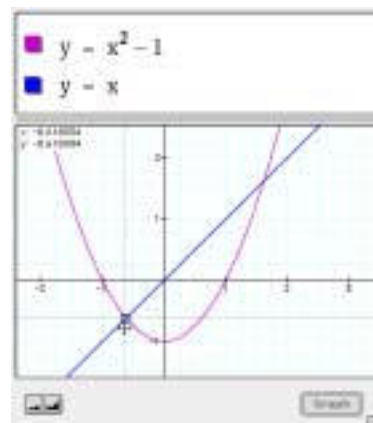
3. $y = \frac{1 + \frac{1}{x}}{x}$

4. $y = \frac{n^{2x}}{1 + \frac{2x+1}{x}}$

Can I trace functions?

I want to know where those two functions intersect.

- ❖ Click on either of the functions. A tracer appears. You'll see the values of x and y in the window, as in the illustration.
- ❖ Drag the tracer. As it moves, the values of x and y update.
- ❖ The tracer will “stick” at intersections. (It will also stick at intercepts and at local maxima and minima.)
- ❖ Click on the other function if you want to trace it instead.



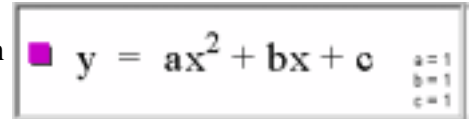
Do I have to use x and y —and n ? Can I use other letters?

Graphing Calculator uses x , y , and z for the axes. You can't change that. There are other special letters as well: r and θ for polar

coordinates, ϕ in spherical coordinates, t for parametric curves, and u and v for parametric surfaces. We'll get to those. But you can use the rest of the letters for constants and parameters. For example:

- ❖ In a new function, enter $y = ax^2 + bx + c$.

A parabola appears. You will also see the values of a , b , and c in the lower-right part of the math pane—the place where the equations live. But let's define those three constants explicitly.



■ $y = ax^2 + bx + c$ $a=1$
 $b=1$
 $c=1$

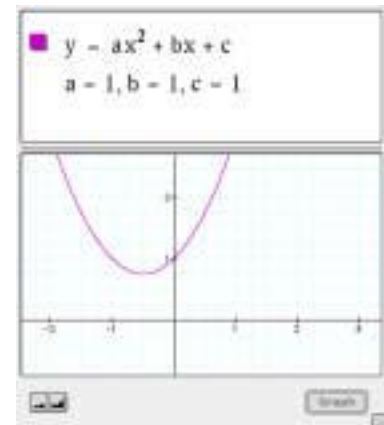
- ❖ Choose **New Math Expression** from the **Math** menu.

- ❖ Type **a=1, b=1, c=1**

Be sure to enter it commas and all.

- ❖ Press **enter** to graph the result.

It will look like the picture. Notice that the line with the constants doesn't have a colored box. It doesn't get graphed—it just tells Graphing Calculator what values to use for a , b , and c .



- ❖ Edit c to be -1 .

- ❖ Press **enter**.

You can see the parabola move down.

- ❖ Change $a = 1$ to be $a = n$.

- ❖ Press **enter** again, and play with the slider to see how changing the parameter a changes the graph.

Things to do

- ❖ For the function $y = ax^2 + bx + c$, describe how the curve changes as you vary each of the parameters a , b , and c .

What should I know about constants and parameters?

In the commercial versions of Graphing Calculator, you can use constants without defining them, that is, without explicitly typing the **a=1, b=1, c=1** line. If you do that, they appear in the lower right of the math pane, by default equal to 1. You can edit them there.

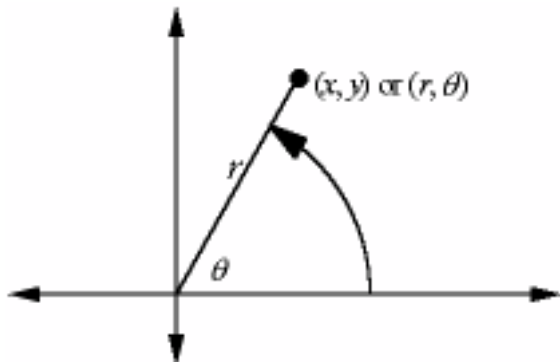
In all versions, you can make a slider for a letter other than n by choosing **Slider Values** from the **Graph** menu and changing the **Variable Name**.

Graphing in Two Dimensions

Graphing in two dimensions includes two more features: polar coordinates and inequalities.

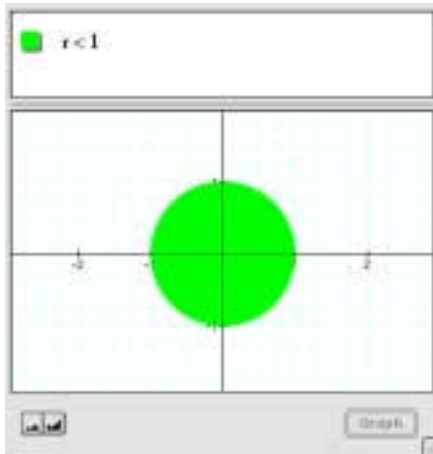
Polar coordinates

Cartesian coordinates—where x goes left to right and y goes bottom to top—are not the only ones. Graphing Calculator also lets you type in *polar coordinates*. Instead of x and y , the coordinates are r (for radius) and θ (theta, an angle); r is the distance from the origin and θ is the angle (in radians, of course) *measured counterclockwise from the positive x -axis*. It's a convention. You just have to learn it.



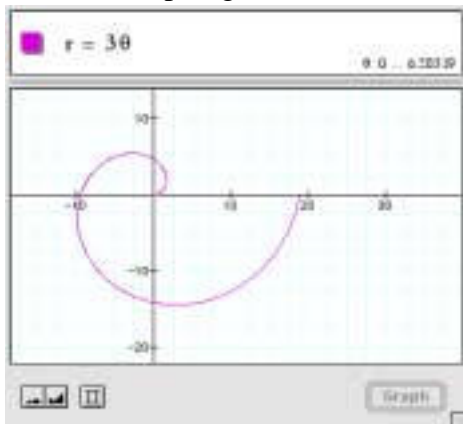
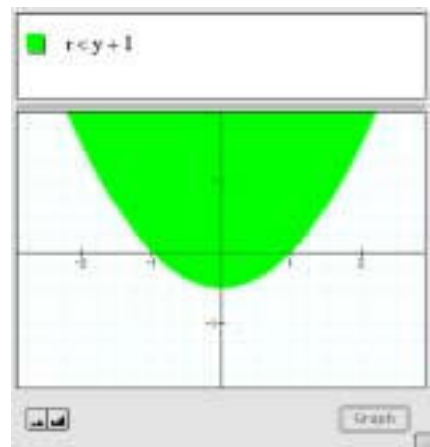
The simplest kind of polar function is a circle, such as the equation $r=1$. If you have a commercial version of Graphing Calculator, you'll see a circle when you type that equation. However, Graphing Calculator 1.3 (the latest free version) can't handle it.

That equation doesn't give a single value of y for each value of x . It isn't a function of x . You can, however, use an inequality.



You can mix r , θ , x , and y in inequalities. See if you can figure out why the graph in the margin looks the way it does. (Sure looks like a parabola, doesn't it?)

What if we add some θ ? Some books and fancier programs will show you terrific functions of r and θ that you can't get in Graphing Calculator 1.3. For example, $r = 3\theta$ looks like this if you use version 3 of Graphing Calculator:



For every value of θ (every angle) between 0 and π , there is a single value of r . It's as if you took $y = 3x$ and wrapped it around a pole, squishing it in at the bottom and stretching it at the top.

Graphing Calculator 1.3 can't do that. To plot a line, Graphing Calculator 1.3 needs y as a function of x (or x as a function of y). But just as we couldn't plot a circle, but could plot its interior, we can plot inequalities using r and θ and that will work.

Things to do

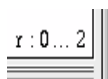
To type a theta, type the word “theta.” In the margin, you can see what it looks like.

- ◆ Try plotting these (type **theta** to get θ):

$$r = \theta \quad r = \frac{\theta}{n} \quad \theta = 16r^2 \quad \theta r^2 = 2 - r$$

Graphing Calculator also does some things you might not expect. Try this:

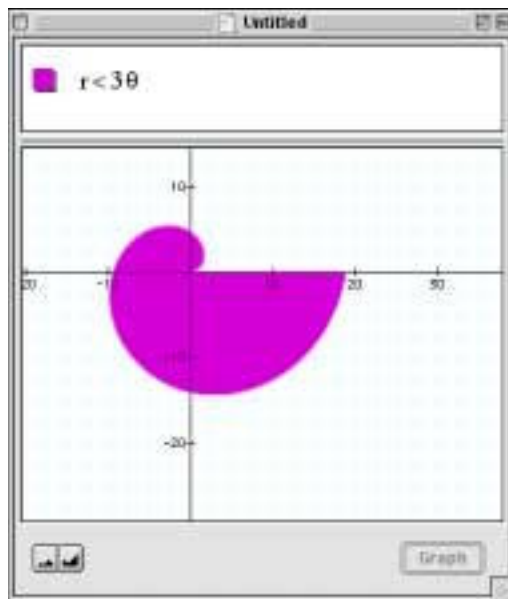
- ◆ Plot $\theta = r^3 - r^2$
- ◆ Look in the lower-right of the math pane. You’ll see this:



It shows the range of r that Graphing Calculator plots. Change it so the range is from 0 to 3. (Change the **2** to **3** and press **enter**.) What changes?

- ◆ Edit the range again, so that r goes from -2 to $+2$, and graph the result.

How do you interpret the result? In particular, how can you have a negative radius?



Inequalities

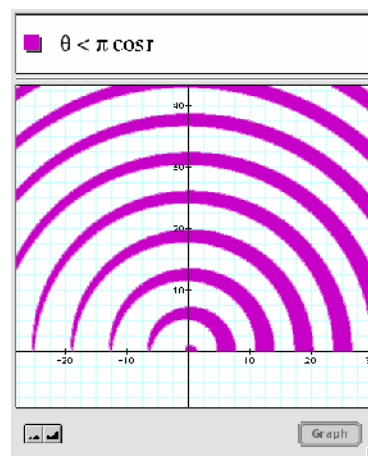
Graphing Calculator will also plot inequalities—many of them amazingly quickly, considering the computation involved.

In some ways, inequalities are more versatile than functions. While functions all have to be in the form $y = \textit{something}$, inequalities can mix up the x and y . Cross terms (ones that include both x and y) make for interesting graphs.

Things to do

- ◆ You should be able to predict these:

$$xy < 0$$



$$\frac{x}{y} > 0$$

- ◆ A couple more that turn out to be straightforward, though they look like they might not be:

$$\sin x < 0$$

$$y \sin x < 0$$

- ◆ Here are some more to try; predicting will get harder and harder:

$$xy < 1$$

$$xy^2 < 0$$

$$x^2 + y^2 < 1$$

$$x^3 + y^3 < 1$$

$$x^4 + y^4 < 1$$

Why do they look like that?

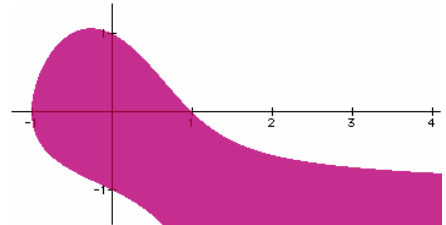
$$x^2 + y^2 + nxy < 1$$

Try $n = 0, 1, 2,$ or $3.$

Examining the shaded results

You can move on to stranger polynomials (this one's shown at the right):

$$x^2 + y^2 + xy + yx^2 < 1$$



As you look at the result, see if you can figure out why it's shaded or not shaded in the different parts of the plane. What about the region just under the x -axis on the right? Why isn't that shaded? That is, what part or parts of the algebraic expression are making it greater than one in that part of the x - y plane?

Also, you can find the curve (though it's not a function) that satisfies

$$x^2 + y^2 + xy + yx^2 = 1$$

It's just the border between the shaded and clear areas.

You've probably noticed that you can "see the computer thinking" as it shades in these inequalities. Here are two similar inequalities. How will they look different? How well does the software perform?

$$|y| < \sqrt{|x|}$$

$$|y| < \sqrt{x}$$

Here's one that takes quite a while if you don't have a fast computer. Can you predict what it will look like?

$$x^y < 1$$

Trigonometric inequalities often produce interesting regions:

$$\cos y > \cos x$$

Regular and straightforward (though maybe unexpected).

$$\csc x < \tan y$$

Looks like somebody's graphic design project.

$$\sin x < \tan y$$

Looks pretty bizarre.

Things to do

Even if you're used to regular functions, it may take a while to develop your intuition about inequalities. Here are a few relatively straightforward ones.

◆ Find the inequalities that produce the following graphs:

<p>1. Remember that the border of an inequality may be a function; just replace the equals sign with $>$ or $<$.</p>		<p>2. Don't forget you can use absolute value.</p>
	<p>3. As above, think about the function.</p>	<p>4. Note that you can use the same transformations with inequalities that we used with functions.</p>

◆ You can make “serial” inequalities as long as they all go in the same direction. Try these:

$$y < x$$

$$1 < r < 2$$

$$\sin(\pi x) > \sin(\pi r y) > -\frac{3}{4}$$

All about circles

Remember, if you have a commercial version of Graphing Calculator, you can also graph circles as regular equations. For more about this, see “Conic sections—using cones!” on page 74.

However, inequalities are useful for everyone, no matter what version you’re using, so let’s take a look at them. The figure in the margin is your basic circle—the interior of it. Here are a few more inequalities to try, all based on $x^2 + y^2 < 1$. See if you can predict how the changes to these expressions affect their graphs.

$$x^2 + y^2 < 2$$

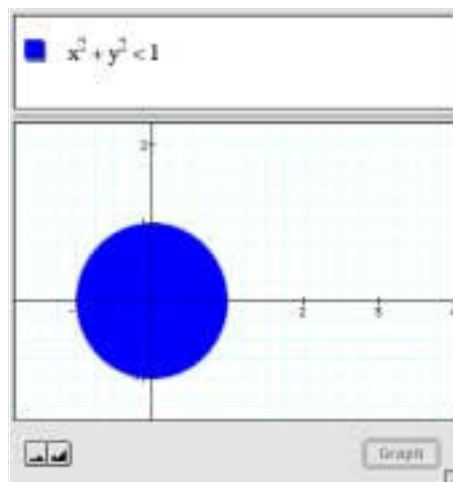
$$x^2 + y^2 - 1 < 1$$

$$(x-1)^2 + y^2 < 1$$

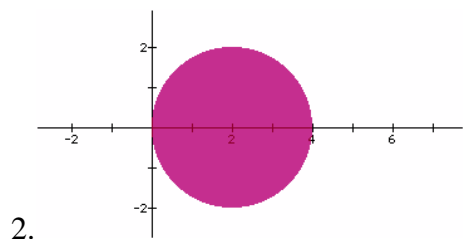
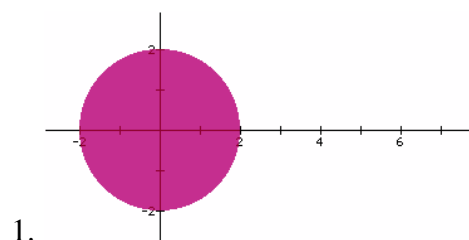
$$(x-1)^2 + (y+2)^2 < 1$$

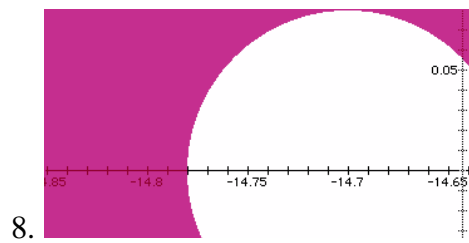
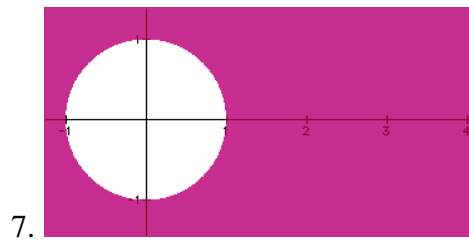
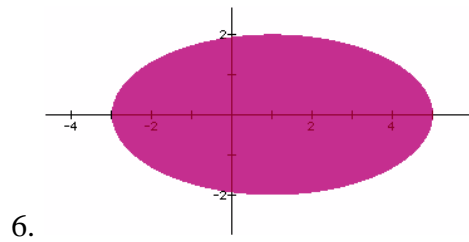
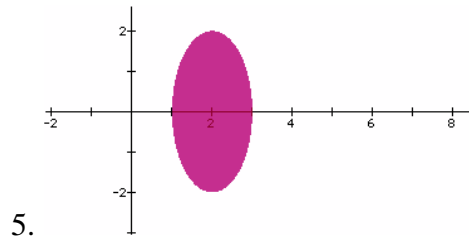
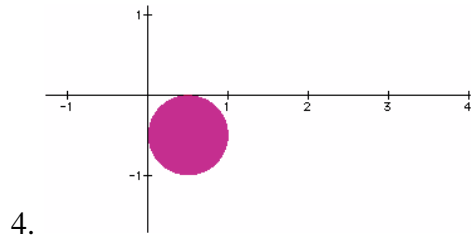
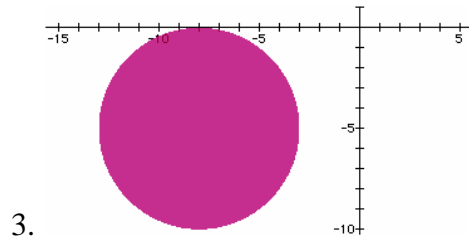
$$x^2 + 2y^2 < 2$$

$$x^2 + y^2 > 2$$



Then try them out. Experiment until you can tell what changing some number will do. Then try to write inequalities that will produce the following graphs:





Other conics and even stranger relations

A circle is a member of a family of curves called *conic sections* because you can get them by slicing up a cone. You get a circle by slicing a cone parallel to its base—perpendicular to its axis. You’ve already seen *ellipses* in the preceding section. You get them from cones by tilting the plane you used to get the circle; an ellipse is just a circle that gets stretched.

Remember, if you have a commercial version of Graphing Calculator, you can also graph conic sections as regular equations. For more about this, see “Conic sections—using cones!” on page 74.

You already know how to get other figures by changing the expressions we used. For example, to make it bigger or smaller, change the number on the right; to get an ellipse, change a coefficient on a term.

Normal circle	Big circle	Squished circle (ellipse)
$x^2 + y^2 < 1$	$x^2 + y^2 < 4$	$x^2 + (2y)^2 < 1$
	Enlarging this number makes the whole figure bigger	This coefficient does the squish in y

What else can you change? Let’s dissect the circle inequality.

You could change an exponent

You could change the plus to a minus

These changes result in (among other things) other conic sections. Here are a few to try:

- | | |
|-----------------|----------------------|
| $x + y^2 < 1$ | A parabola |
| $x^2 + y < 1$ | Another parabola |
| $x^2 - y < 1$ | Yet another parabola |
| $x^2 - y^2 < 1$ | A hyperbola |

$$y^2 - x^2 < 1$$

Another hyperbola

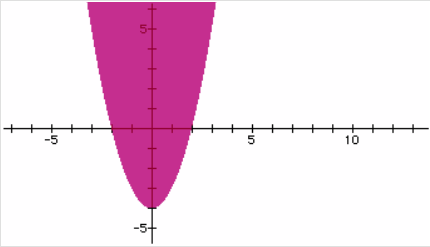
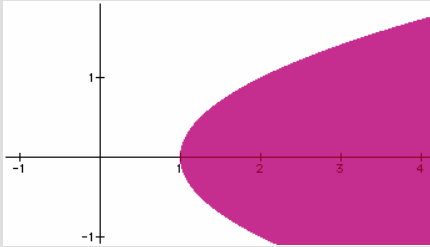
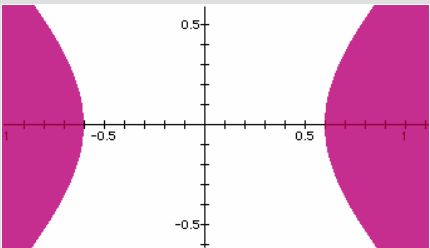
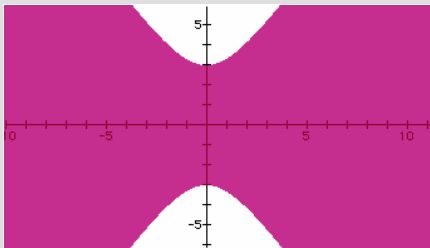
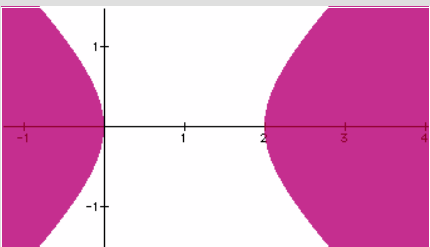
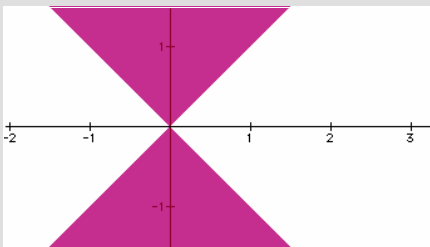
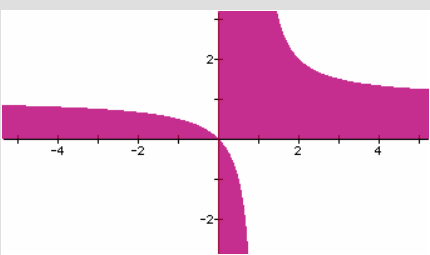
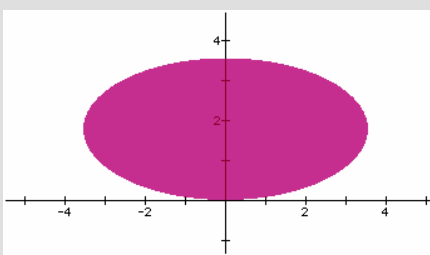
$$y^2 - x^3 < 1$$

Something completely different

You can also change the global expanding and squishing coefficients—and you can slide these things all over the Cartesian plane.

Things to do

Using all you've done with circles and the relations above, see if you can figure out what inequalities produced these graphs:

<p>1. This is a parabola.</p>			<p>2. So is this.</p>
<p>3. This is a hyperbola.</p>			<p>4. This one's axes and shading are flipped.</p>
<p>5. This is a hyperbola similar to the one above but not centered on the origin.</p>			<p>6. This is a hyperbola, too, a special one. What slice do you take of cones to get this shape?</p>
<p>7. This is not a conic section. Experiment with the hyperbola inequality, but change the exponents (use n if you wish).</p>			<p>8. An elliptical review.</p>

- ◆ Inequalities using r and θ are worth exploring. Here are more inequalities to try:

$$r < \sin 4\theta$$

$$0 < 1 - (r - 2)^2$$

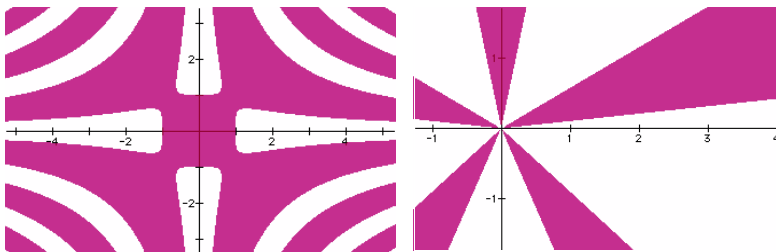
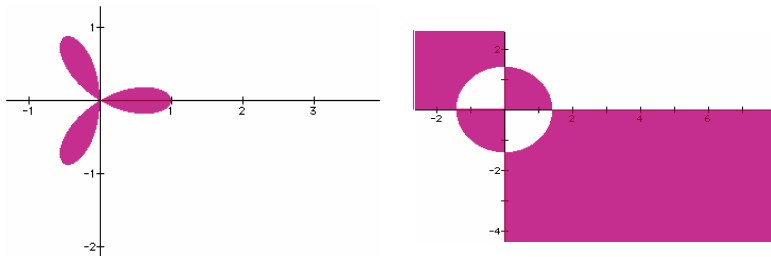
$$1 > \left| \frac{r}{\theta} \right|$$

$$\sin \frac{1}{r} < 0.1$$

$$\sin(xy) < \frac{1}{r}$$

$$xy < \sin(n\theta)$$

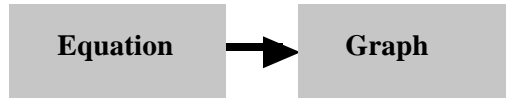
- ◆ As with the other inequalities, try to figure out why the white regions are white and the shaded are shaded. Then, using your experience to help you, see if you can find inequalities that produce the following graphs (or at least something close):



Transformations

In traditional classes on functions—usually algebra at some level—you make lots of graphs, first by plotting points and connecting them, and later by using a formula to tell how to draw the function. For example, if we see $y = 65x - 120$, we see that 65 is the slope and -120 is the y -intercept, and we draw the line accordingly.

Schematically,

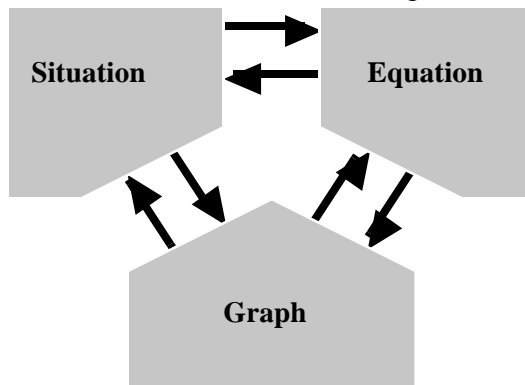


But in real life, there’s more to it. You have a situation—a moving car, say. From that, you want to make some conclusions, for example, a prediction about where it will be. Maybe you draw a graph of how the car moves and you use it to create an equation or formula. Then, you use the formula to figure out what you want to know.

Schematically,



In fact, we want to be able to translate among all three, like this:



The traditional approach addresses only one of the arrows. Here, we’ll discuss its reverse—how do we figure out an equation if we know the graph? This is harder than graphing an equation, much as writing music is harder than playing it.¹

1. There are ways in which playing is harder than writing. The point is to be able to do both.

We will use transformations to help us in this endeavor. After all, a wide variety of functions are really just transformations of a few basic forms. We'll use Graphing Calculator to see how to change mathematical expressions so that the graphs stretch, shrink, slide, and combine. In this way, we'll learn how to look at a graph and figure out its equation.

Here are a few kinds of transformations:

- Translations are the simplest. A translation moves a curve left or right, up or down, without changing its shape.
- Dilations are next. Use a dilation to squish, stretch, and flip functions in the x-direction, the y-direction, or both.
- Inversions are produced by using the inverse of a function, which amounts to flipping the axes. This is how you get logarithms—they're the inverse of exponentials.
- Reciprocals are produced by taking one-over-whatever, as in .

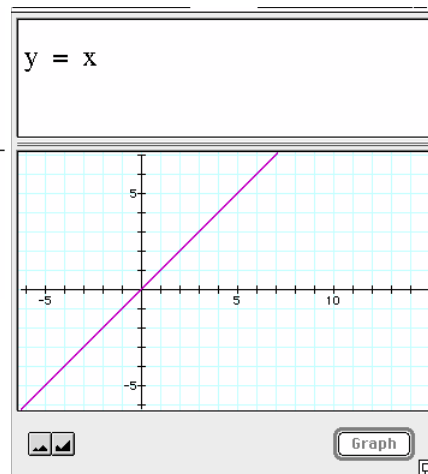
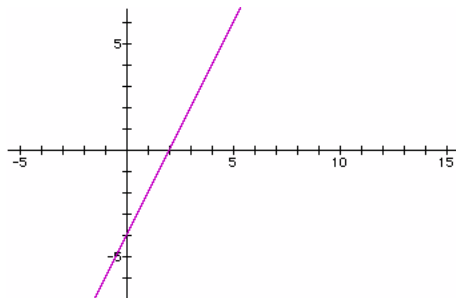
In this section, we'll explore the first two. We'll come back to the others soon. Later on, we'll also see what happens when you add functions, multiply them, and compose them.

Translations, dilations, and $y = mx + b$

Let's start with that bread-and-butter function, the straight line— $y = mx + b$. This equation can take many forms. So there may be more than one way to get the equation of the line you want.

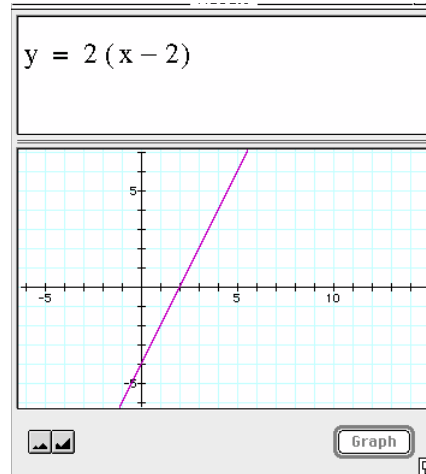
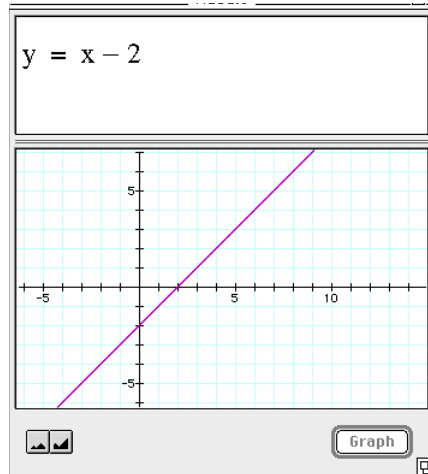
1. The traditional, math-class way to get the equation for the line at right is to say, "the slope is rise over run, which is 4 over 2, so the slope is 2. The intercept is -4. So $y = 2x - 4$."

That works, but let's see how to do this with transformations.



2. We start with the line $y = x$. Now we slide it and stretch it to get the line we want.

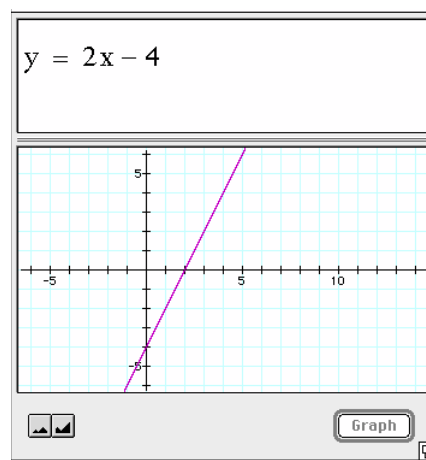
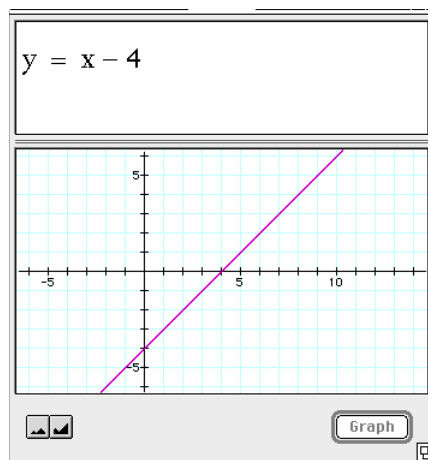
3. Suppose we slide it to the right 2. That would be subtracting 2 from every x . Now the x -intercept is correct.



4. It's not steep enough, so we stretch it in the y -direction by multiplying the whole right-hand side by the stretch factor. We want every point on the line to be twice as far from the axis as it is now, so we multiply by 2.

The end result, of course, is exactly the same thing as the slope-intercept equation, $y = 2x - 4$. There is another way to accomplish the same thing. Here are “replacement” frames for 3 and 4 above:

3a. Suppose we slide it down 4. That would be subtracting 4 from the whole expression. Now the y -intercept is correct.



4a. It's not steep enough, so we squish it in the x -direction by multiplying every x by the squish factor. We want every point on the line to be twice as close to the axis as it is now, so we multiply by 2.

So there are two ways to get to the same place—by stretching in y and by squishing in x . It's hard to see.

Things to do

- ◆ Set up the equation in the example in the margin and then change the value of n by dragging the slider knob. Play the animation, too. What does the n do to the curve? What if n is positive? Which way does the curve move? What if n is negative?
- ◆ Now we'll look at a different situation. Set up the window as before, but this time, instead of $y = x^2 + n$, use $y = (x + n)^2$.

Answer the same questions: What does the n do? And again, what if n is positive? What if it's negative?

- ◆ Let's try another function and see if we can slide it back and forth. Try these expressions:

$$y = \sin x \quad y = \sin(x + 1) \quad y = \sin(x + n)$$

Again, see what happens when n is positive and negative. How would you change the function to make it slide up and down?

- ◆ What's the rule for making a function slide up and down?
- ◆ What's the rule for making a function slide back and forth?

However...

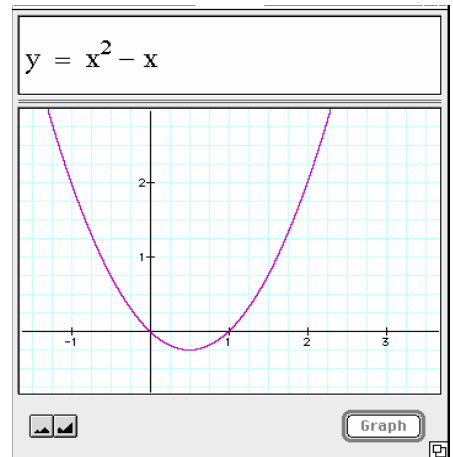
What if there's more than one x in an expression? Try it.

- ◆ First graph $y = x^2 - x$ (as shown in the margin).
- ◆ Then see if you can slide it back and forth. Try these expressions:

$$y = (x^2 - n) - x$$

$$y = x^2 - (x - n)$$

$$y = (x - n)^2 - (x - n)$$



The lessons should be:

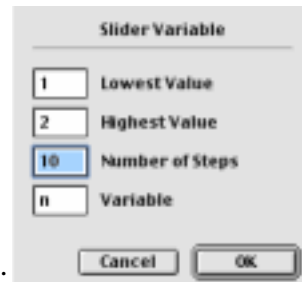
- ◆ To make a curve slide to the right, subtract the distance you want it to slide from *every* x in the expression.
- ◆ To make a curve slide up, add the distance you want to slide it to the *entire* expression.

Dilations: stretching, squishing, and flipping

Dilations are transformations that squish or stretch curves. On this page, we'll learn how to squish and stretch the sine function vertically and horizontally.

First,

- ◆ Enter $y = \sin(nx)$.
- ◆ Change the slider values to match the illustration in the margin.



Your job is to figure out what n does as fully as possible. Move the slider. Play the animation. At first we set it up so that you look only at positive values of n . But you can change that.

Here are some questions you can explore:

- ❖ What happens to the curve as n gets bigger?
- ❖ What happens to the curve if n is negative?

Now let's look at a different version of the sine function.

- ❖ Enter $y = \sin(nx)$.
- ❖ Do exactly the same exploration. When you understand the sine function, try a different function, such as $y = x^2$ or $y = \cos x$.

Things to do

- ❖ What's the rule for stretching a function horizontally?
- ❖ What's the rule for squishing a function horizontally?
- ❖ What's the rule for stretching a function vertically?
- ❖ What's the rule for squishing a function vertically?

One lesson should be the following: To affect a curve *horizontally*, you have to act upon *every* x in the function; to affect it *vertically*, you have to act upon the whole expression (which amounts to doing something to every y , if you want to look at it that way).

Another observation: Addition causes translation; multiplication causes dilation.

Putting transformations together

Let's put all three of the transformations we introduced above into one equation.

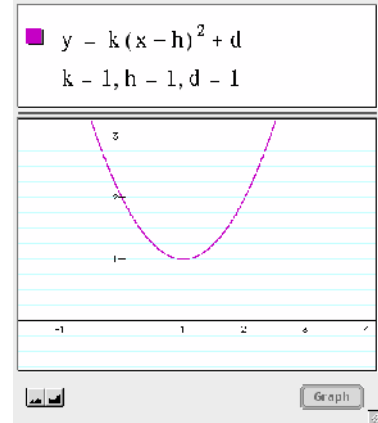
◆ Edit the function to read $y = k(x - h)^2 + d$.

◆ Make a **New Math Expression**.

◆ Assign values to the parameters.

That is, type in $k=1$, $h=1$, $d=1$ —as in the picture at right.

◆ Graph the result.



Things to do

◆ Play with the values of k , h , and d by editing the numbers in the math pane or by assigning them (one at a time) to the slider (one good way is to set $k = n$) and dragging the slider around.

Then do the following:

◆ For each parameter (k , h , and d), tell what kind of transformation it causes: a translation or a dilation. If it's a translation, tell which direction; if it's a dilation, tell how the number relates to the shape.

◆ Explain why it's “ $-h$ ” instead of “ $+h$ ” in the formula above.

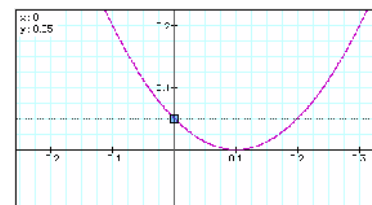
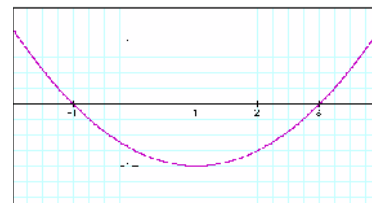
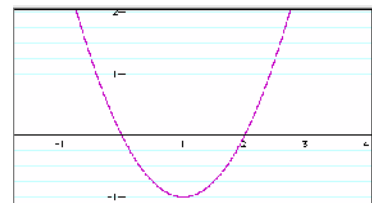
◆ What happens when each of the parameters changes from positive to negative or vice versa? Explain why it does that—and why the dilation is so different from the translations.

◆ Find a function that matches each of the pictures in the right-hand column. It will probably be easiest to change k , h , and d (now that you know what they do) and just write down the result. That is, if you find that $k = 2$, $h = -1$, and $d = 1$ match the graph, your function would be

$$y = 2(x + 1) + 1$$

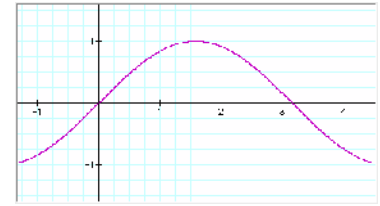
Note: Sometimes it's hard to edit the parameters. Here's a trick: if you're editing one of them, and you want to change another, press **tab**. That will move you to the next value. Then, when you want to graph the result, press **enter**. This way, you can try different sets of parameters in rapid succession.

(Solutions on page 144)



Once more, with waves

We've learned about transforming parabolas. What if we transform a different function? We could pick any function (higher-order polynomials, exponentials, whatever) but the most instructive is a trigonometric function—a periodic wave. We'll start with $y = \sin(x)$, shown at right. Remember: x is in radians.



Once again, we're going to look at how changing a parameter changes how a graph looks. Again, we'll be looking for two transformations: translations and dilations.

- ◆ Start with a fresh Graphing Calculator document. (Choose **New** from the **File** menu.)

- ◆ Enter each of functions below in turn.

Again, have the parameter n run from -1 to $+1$.

$$y = n \sin(x) \qquad y = \sin(nx)$$

$$y = \sin(x + n) \qquad y = \sin(x) + n$$

- ◆ For each function, describe how changing n changes the graph:

As we did with the parabola, we put the four transformations together.

- ◆ Enter the following and play with the different parameters so you're sure you know what they do:

$$y = A \sin(f(x - h)) + d$$

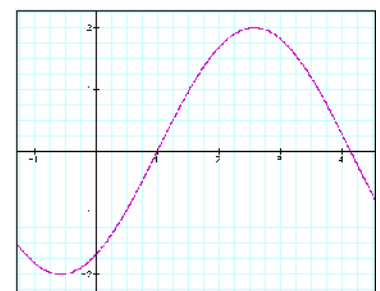
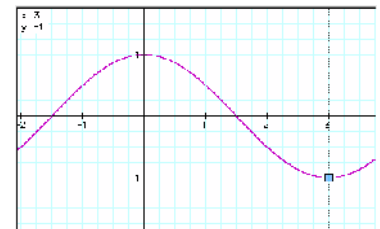
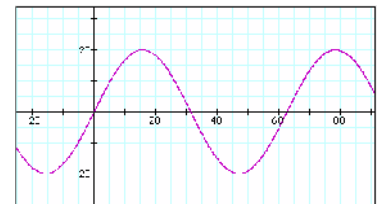
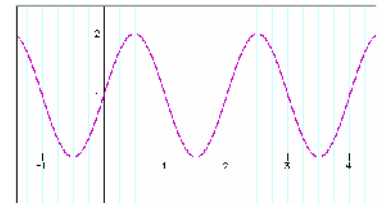
Now we have four parameters.

- ◆ For each parameter (A , f , h , and d), tell whether it causes a translation or a dilation. If it's a translation, tell which direction; if it's a dilation, tell how the number relates to the shape.

Things to do

- ◆ Find a function that matches each of the pictures at right. It may be easiest to change A , f , h , and d (now that you know what they do) and write down the result. (Solutions on page 144.)

Note: For some of these, it will be easy to use a slider; for others, you'll probably need to type values in the math pane. If you don't see your parameter in the math pane, it may currently be a slider.

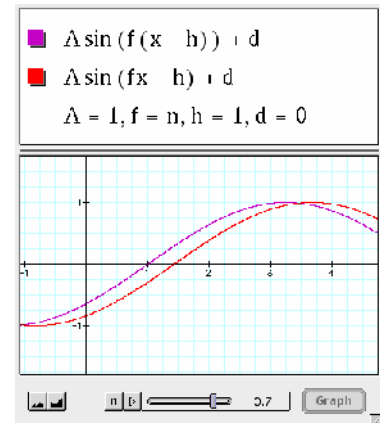


Choosing a parameterization

Having done these exercises—both with the parabola and with the sine wave—we should notice that there are several ways we could have chosen to write the functions.

You probably noticed, for example, that the f term in the sine stretches or squishes—dilates—the function horizontally. And that makes some sense as it multiplies the x in the function. But why didn't we multiply f by x before we subtracted h ?

We could have! I find it easier to build the functions I want using the particular form I gave you. But don't take my word for it; after all, it may be easier for you using a different form.



Things to do

Compare the two forms below using Graphing Calculator. You can even put them on the same graph at the same time, as shown in the illustration. Then change the parameters that are functionally different (h and f in the first example) to see how changes in these parameters affect the functions. Finally, write up what you found out. Especially, decide which you like better and why.

❖ Compare $y = A \sin(f(x-h)) + d$ with $A \sin(fx-h) + d$

❖ Compare $y = A \sin(f(x-h)) + d$ with

$$y = A \sin\left(\left(\frac{x-h}{f}\right) + d\right)$$

❖ Compare $y = A \sin(f(x-h)) + d$ with

$$y = A \sin(2\pi f(x-h)) + d. \text{ (Note: Just type "pi" to get the } \pi \text{ character.)}$$

❖ Compare $y = k(x-h)^2 + d$ with $y = (kx-h)^2 + d$

Translations, the minus sign, and symmetry

Here's another thing that you, being alert, may have puzzled about. We'll use it to open the door on Graphing Calculator's algebra-processing capabilities.

You probably noticed that when you do a translation, you use d to do a vertical translation and h to do a horizontal translation. But to

get a translation in the positive direction, you add d but subtract h . What gives?

For me, the d is easy: When you add the d , because of the way we put the d into the function, you add it to the whole thing. So whatever value the rest of the function gives us, we add d to it before we plot. That just moves the whole graph up by that amount.

The h is tougher. If I'm using a sine, for example, I think to myself, look, the function crosses the x -axis going up when the argument is zero (e.g., so we get $y = A \sin(0)$). If I want to move that spot to the right—to $x = 1$, say—I have to make it so that when $x = 1$, the argument is zero. So I have to subtract the one from the x in the function. That is, I use $y = A \sin(x - 1)$. That way, when $x = 1$, the argument—the thing in parentheses—is zero, and the function moves to the right.

The key thing about horizontal translations is that you have to subtract the amount—the h —from every x in the function. Enter this function into Graphing Calculator:

$$y = \frac{\sin(\pi x)}{\pi x}$$

(We could have used plain old $y = \sin x/x$, but using pi makes it scale well for default window settings, and makes the intercepts into integers.)

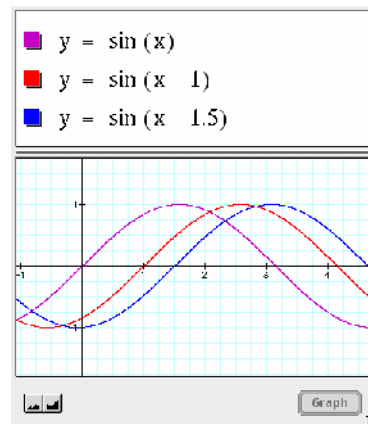
Compare what you see with the following three alternatives. Make h into a slider to get the full effect:

$$y = \frac{\sin(\pi(x-h))}{\pi x} \quad y = \frac{\sin(\pi x)}{\pi(x-h)} \quad y = \frac{\sin(\pi(x-h))}{\pi(x-h)}$$

Note: What's the best way to compare? Depending on your preferences, you could put them all on the same graph, open three new windows, or just look at them in sequence, one at a time.

The difference between horizontal and vertical

If you now believe, deep in your heart, that subtraction is best for moving a function to the right, but that it makes perfect sense to use addition when you move a function up, your sense of mathematical aesthetics could rebel. Let's try to set things right.



- ◆ Enter this as a function: $y = (x - 2)^2 + 1$. This is a parabola, scooted right two and up one.

Note: If your parabola is up in the corner of your window, drag the axes to move it more where you want it.

Now you get to use *drag algebra*. (If you don't know what drag algebra is, see "What about simple algebra?" on page 13.)

- ◆ In the math pane—the top part where the function is—select the 1 and move it to the other side of the equals sign.

Shazam! The one changes sign! But because you changed the function, the graph vanished.

- ◆ Press **enter** to make the graph reappear.

It's identical to the original one, right? That's because *when you drag parts of an equation, Graphing Calculator preserves algebraic equality*. This has stunning and sometimes unexpected consequences.

Most important in this context, however, is that we can now see how the two axes really are the same. That is,

$$y = (x - 2)^2 + 1 \text{ is the same as } (y - 1) = (x - 2)^2$$

Both axes have subtraction. Both translate in the positive direction.

Note: Another way to look at this translation thing is that you're not moving the function, you're moving the *axes*. The actual function was $y = x^2$, but then we moved the *y*-axis down one and the *x*-axis left two. Some people prefer this way of thinking about it.

Just as when we were discussing translations in *x*, if you have an expression that has more than one *y*, you have to subtract the translation from every *y* in the expression. Alter the following equations so they have a slider parameter *d* that translates the graph in the vertical direction:

$$x = y^2 \quad x^2 + y^2 = 1 \quad \frac{\sin(\pi x)}{\sin(\pi y)} = xy$$

$$2 - x - y - 2xy + y^2 = 0$$

$$y = (x - 2)^2 + 1$$

$$y - 1 = (x - 2)^2$$

$$y - 1 = (x - 2)^2$$

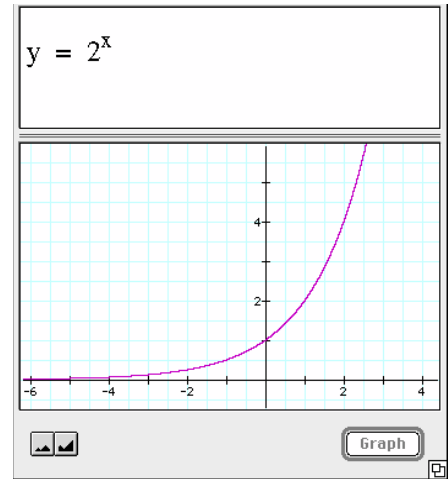
Things to do

Special task if this was all too easy:

The last expression above graphs as a parabola. But if you change the “2”—and not by much—you can change it into an ellipse or a hyperbola.

- ◆ At what values of that parameter does the conic change from one form to another, and why?

Explain it as fully as you can. Ha. That’ll keep you busy...
(Solution on page 144.)

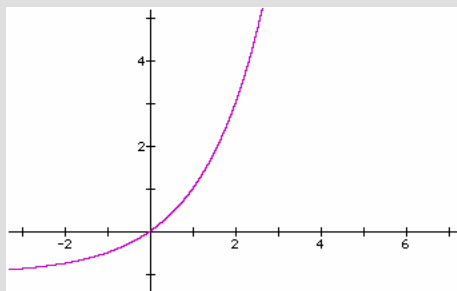


Exponentials

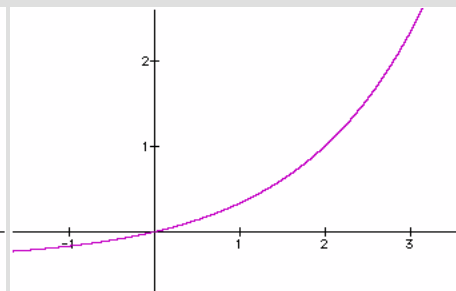
These functions are something-to-the- x -power. On this page, the “something” is 2, like the one in the margin.

You’ll have to flip, slide, squish, and so forth to get that one to match these. What are the equations of the curves below?

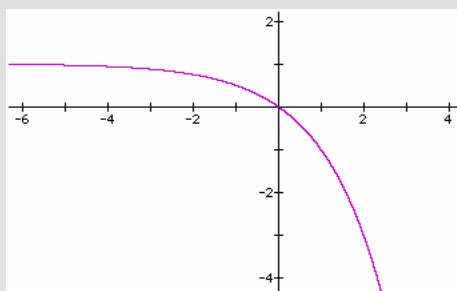
1.
This is almost like the basic exponential above.



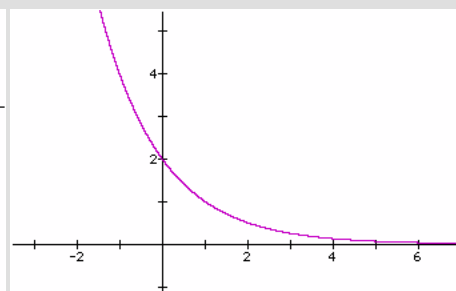
2.
This curve passes through (2, 1) as well as the origin.



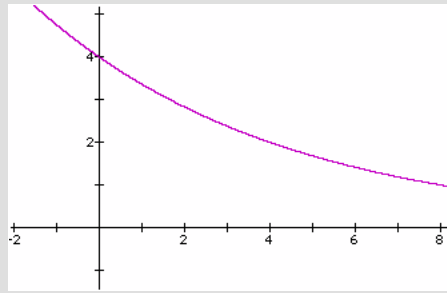
3.
This is like the one above but flipped over. How do you do a vertical flip?



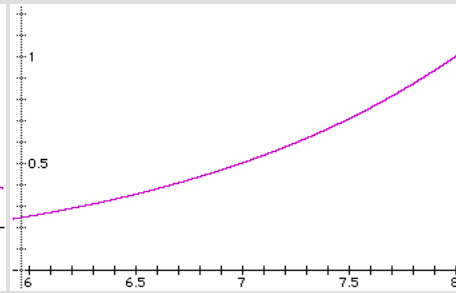
4.
This one passes through (-1, 4).



5. This decaying function passes through (8, 1) and (4, 2).



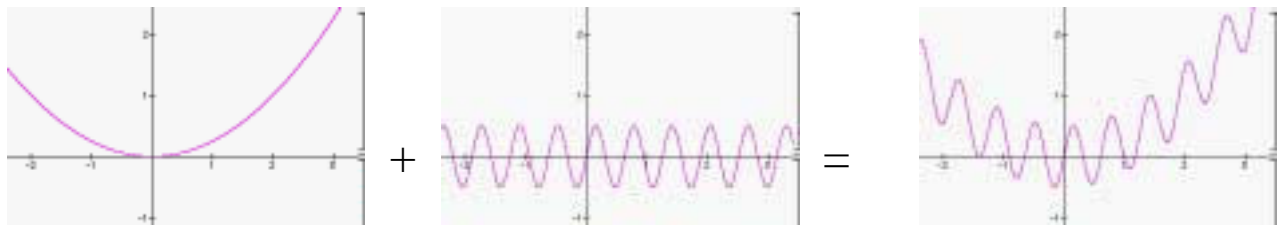
6. This curve doubles between 6 and 7, and again between 7 and 8.



Adding functions part I

When you cannot make your graph look the way you want it to by sliding or stretching functions you know, sometimes adding or multiplying them is the way to go. We'll start with adding.

Adding functions can be simple. Add a sine wave to a parabola:



$$y = \frac{x^2}{4} + y = \frac{1}{2} \sin(10x) = y = \frac{x^2}{4} + \frac{1}{2} \sin(10x)$$

We've just added the wiggly sine to the gently curving parabola. You can use Graphing Calculator to help make that clear. Use the slider parameter n to determine how much sine to add:

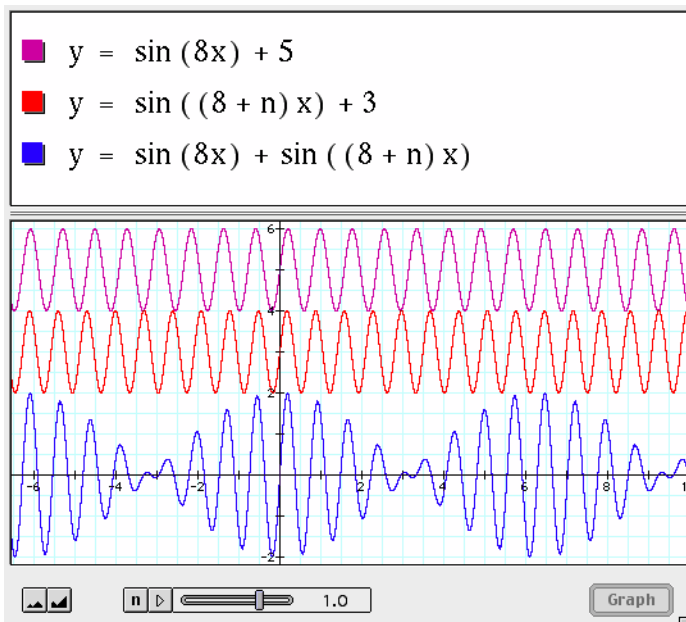
- ◆ Start with a new document
- ◆ Enter $y = \frac{x^2}{4} + \frac{1}{2} \sin(10x)$ and graph the result.
- ◆ Edit the formula, substituting n for the $1/2$ coefficient
- ◆ Change the range of the slider (the n button at the bottom or **Slider Values** in the **Graph** menu) to go from -1 to $+1$ in 20 steps.
- ◆ Animate!

Things to do

We viewed this addition as changing the parabola by adding a sine—but you can look at it as altering the sine by adding a parabola. Change the position of the n in the equation to demonstrate that. Which makes more sense to you?

Adding functions part II

Sometimes it’s harder to see what’s going on when you add two functions. One really great example is with beats—when you add two sinusoids whose frequencies are very close.



In this case, it’s good to be able to see both functions and their sum at once. In the illustration, we’ve used a vertical translation to let us see them separately, and a parameter n to control one of the frequencies.

So create this picture for yourself, and play with the slider. This slider is set to run from -2 to $+2$ in 40 steps:

Things to do

Here are some tasks that may help you understand what’s going on:

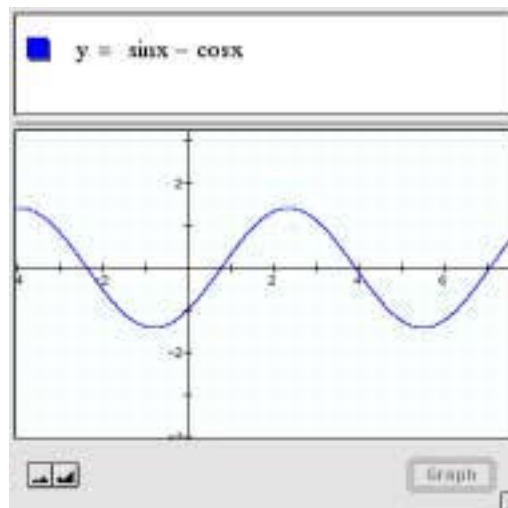
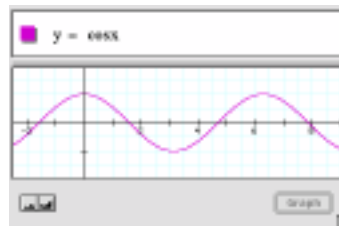
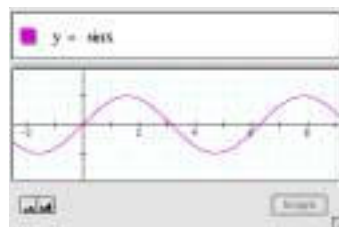
- ◆ Instead of keeping the two top waves separate, put them together (change the $+5$ to a $+3$). Is it easier (or harder) to see where the two waves are in phase, and constructively interfere?

- ◆ Change the slider to give you looser control. Make it run from -10 to $+10$ in 40 steps. How does that effect how well you can tell what's happening with these functions? What's going on when n is close to -8 ?

Difference functions

You can also make a function that is the distance between two curves. That distance is just the *difference* between the two functions—so you subtract instead of add.

Here we look at the two functions, $y = \sin x$ and $y = \cos x$.



This function shows the distance between them. Note that the distance can be negative—where the cosine is above the sine. We could have used absolute value to get the absolute difference.

Things to do

- ◆ Can you find a function of the form $y = A\sin(x-b)$ that matches the difference function above?

Hint 1: It will have a π in it.

Hint 2: Look at the amplitude of that wave. How big does y get? It looks like about 1.4. (It's a little more.) That's close to the square root of 2. Coincidence? I don't think so....

- ◆ Study the difference between e^x and e^{-x} . The result looks a little like tangent, a little like x^3 , but it's not. Use the keypad and try the hyperbolic functions. Find out what hyperbolic function is equal to $e^x - e^{-x}$. You will have to dilate it in the y -direction to make it fit exactly.
- ◆ Study the difference between $\sin^2 x$ and $\cos^2 x$. As before, find the difference function.

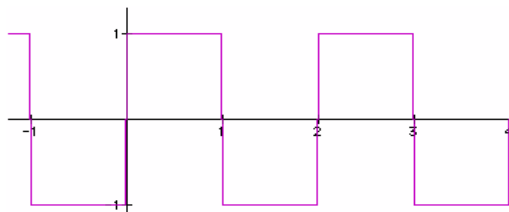
Fourier series

Fourier series are made up of sinusoids, all of which have frequencies that are integer multiples of some fundamental frequency. The trick, as with Taylor series, is to figure out what the coefficients are. In summation notation, we say (for odd functions of period 2π , but that's just being picky in this context):

$$F(x) = \sum_{k=1}^{\infty} a_k \sin(kx) = a_1 \sin(x) + a_2 \sin(2x) + \dots$$

...and the trick is finding the coefficients a_k .

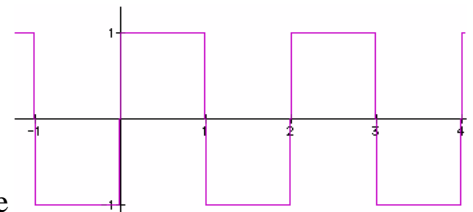
You can find those coefficients by using calculus on complex exponentials, or you can use Graphing Calculator and just build your function out of sines.



Example: Making Square Waves

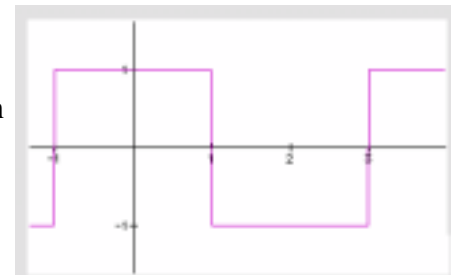
Let's try to approximate a square wave (shown at right) with a period of 2. One way to do it appears on the next page.

A great thing about using Fourier series on periodic functions is that the first few terms often are a pretty good approximation to the whole function, not just the region around a special point.



Things to do

- ◆ Why are all of the even coefficients zero in the analysis above?
- ◆ Suppose you had a square wave that looked like the one in the margin. Make a Fourier series (of at least three terms) to approximate it. You'll soon see that sines don't help. Why not?
- ◆ Look at the square wave function written in summation notation on page 50. Explain where the $(2k + 1)$ terms come from.
- ◆ In the Fourier series for the square wave, the coefficient decreased (in this case, from 1 to $1/3$ to $1/5$) as we added more terms. That's typical of Fourier series in general. Why do you think Fourier coefficients tend to get smaller the further you get in the series?

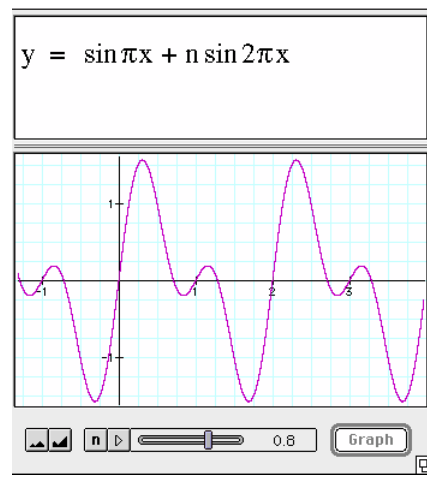
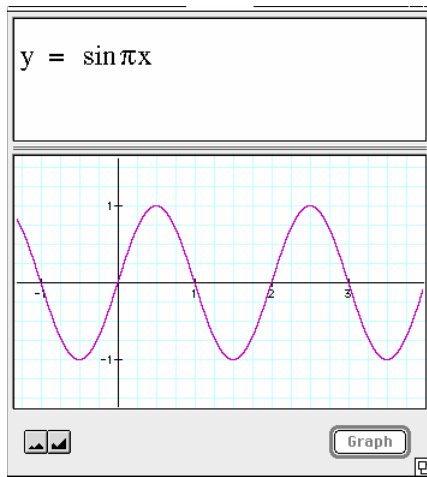


- ◆ Try building up $f(x) = \cos x + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \dots$
- ◆ What shape will it have when you carry the series out to infinity?

Fourier series are used extensively in engineering, especially for processing images and other signals. Finding the coefficients of a Fourier series is the same as doing a spectral analysis of a function.

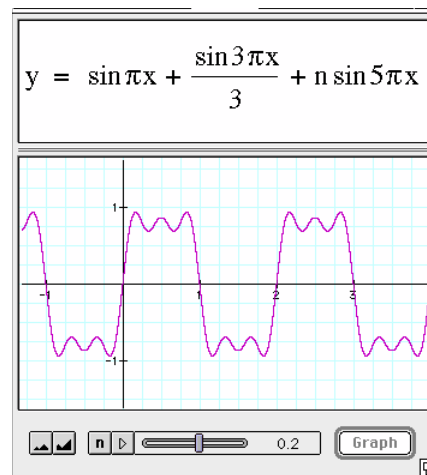
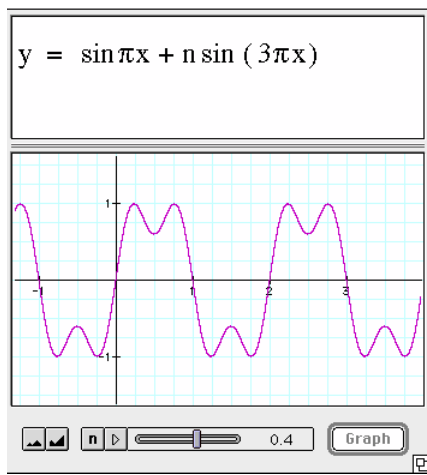
The Square Wave in Detail

Start with a sine wave (shown at right). We have to give it the right period, and by now we've learned that doing a shrink by a factor of π makes the function pass through $(1, 0)$.



Ty a term with a frequency twice that of the first. We don't know the coefficient, so use n with a range from 0 to 2. Sadly, nothing helps. It's OK to have a coefficient of zero, so we move on to the third component.

Here things are looking better. The value of 0.4 was pretty good, but .33 might be better.—substitute the value as one-third. Go on to the fourth component (which looks terrible, so we assume its coefficient is zero) and then the fifth.



Pretty good—and 0.2 (the value of n in the picture) is about one-fifth. Now we're getting somewhere, and a pattern emerges. But all the even coefficients are zero. On some reflection, that makes sense. Why?

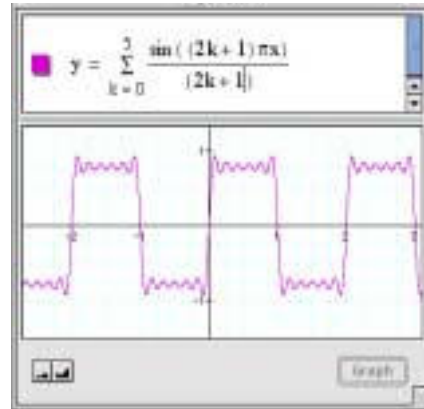
The pattern suggests that the square wave function is

$$\text{Square}(x) = \sin \pi x + \frac{\sin 3 \pi x}{3} + \frac{\sin 5 \pi x}{5} + \frac{\sin 7 \pi x}{7} + \dots$$

which turns out to be correct. We can write this using summation notation:

$$\text{Square}(x) = \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi x}{(2k+1)}$$

You make summations with the big sigma by choosing **Summation** from the **Math** menu. Edit them normally.

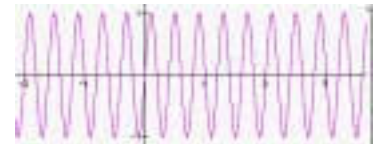


Multiplying functions: envelopes

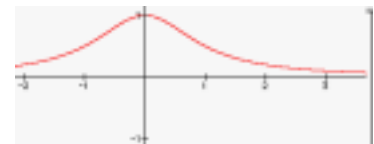
Multiplying two functions can be complicated. But one situation that's easy to understand visually is when one of the functions is usually between zero and one, and the other one is doing all kinds of stuff (e.g., a wave). In that case, the first acts as an *envelope* for the second.

For example, you can make a wave packet.

◆ Take this wave ($y = \sin(5\pi x)$):



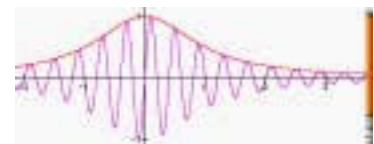
◆ And multiply it by this envelope ($y = \frac{1}{1+x^2}$):

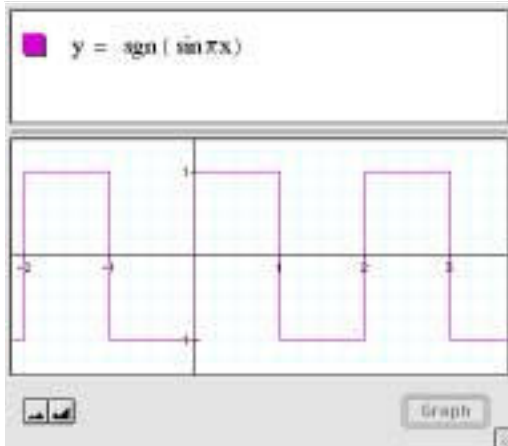


◆ To get this result—a wave packet:



◆ If you plot the result and the envelope together, you can see how well they fit:





The signum function is great in composition and multiplications; it has a value of 1 when $x > 0$ and -1 if $x < 0$. For more about this function, see “Signum: a Useful Function to Know About” on page 103.

Things to do

- ◆ See if you can make a graph like the one at right:
- ◆ Make a wave packet that moves across the screen as you change a slider parameter.
- ◆ Look back at the “beats” picture on page 46. It’s the sum of two fairly high-frequency waves. But you could see that as a high-frequency wave with an envelope that’s a low-frequency wave. Figure out the frequency of the low-frequency wave. What does it depend on? If possible, use that insight to derive a trigonometric identity about the addition of two sines.



(Solutions on page 145)

Reciprocals of functions



Another way to modify functions is to take their reciprocals. One way to do that is to select the part you want the reciprocal of, and then use the $1/x$ button on the keypad. Try this out with the basic functions and their inverses. Three samples appear in the margin.

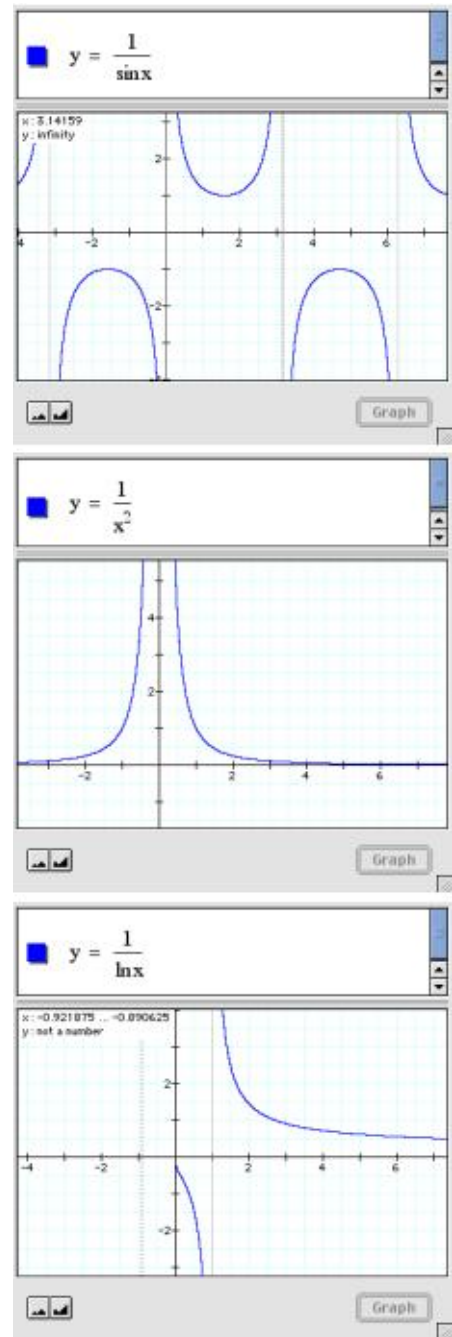
You might have noticed that they explode. Why? Because we finally have a denominator, and when that denominator goes to zero, the whole function blows up.

This characteristic explosive behavior is worth exploring. Graph the functions below, and when they explode, be sure you know why. Some of them have places where you might expect them to blow up, but they don't. In those cases, why not?

- | | |
|--|--|
| <p>◆ $y = \frac{x+1}{x-1}$</p> <p>◆ $y = \frac{2^x + 1}{2^x - 1}$</p> <p>◆ $y = \frac{x^2 - 2x + 1}{x - 1}$</p> <p>◆ $y = \frac{\sin x}{\tan x}$</p> | <p>$y = \frac{2^{x+1}}{2^{x-1}}$</p> <p>$y = \frac{x^2 + 2x + 1}{x - 1}$</p> <p>$y = \frac{\cos x}{\tan x}$</p> <p>$z = \frac{1}{r}$</p> |
|--|--|

Things to do

- ◆ Without using Graphing Calculator, can you tell where $y = 4x(x + 1)(x - 3)$ crosses the x -axis? Verify that using the software.



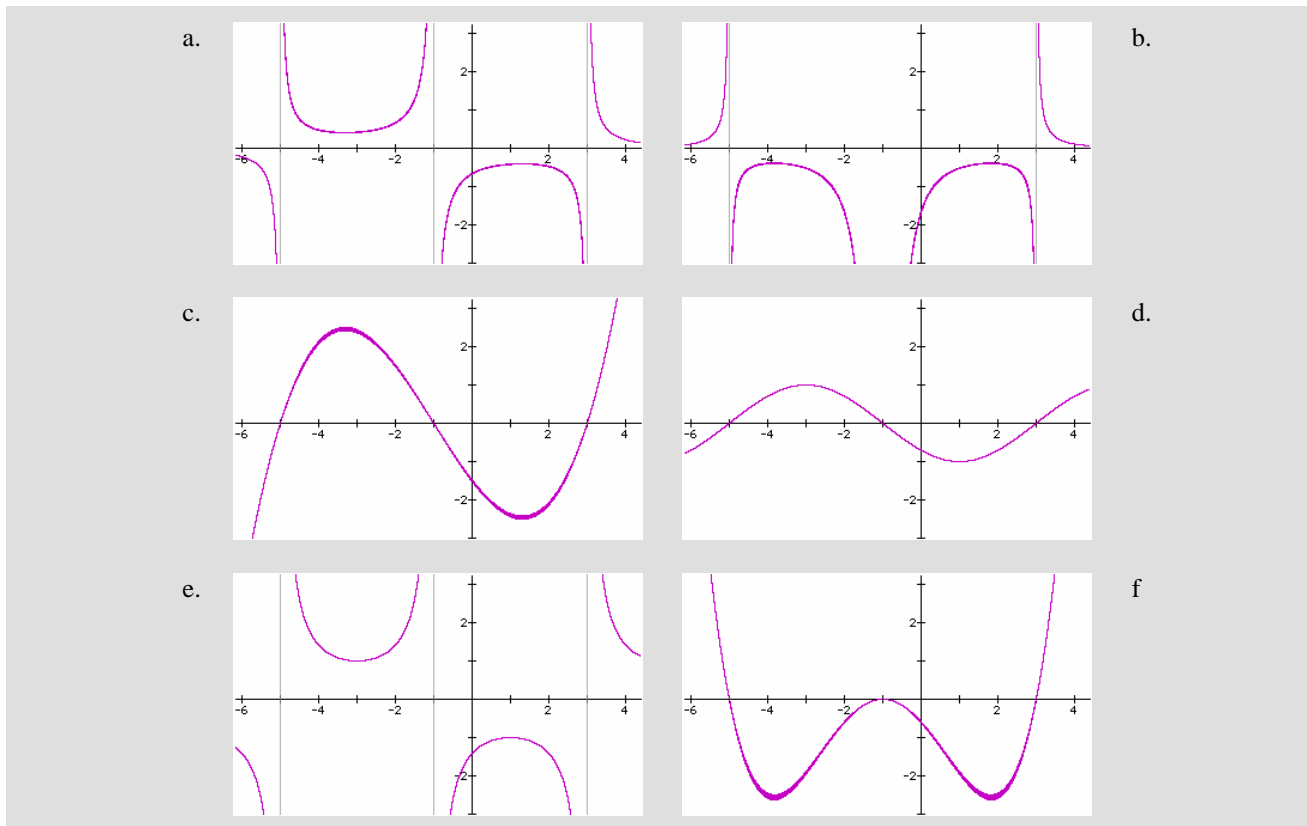
Three inverses of common functions. Notice how Graphing Calculator handles the undefined values in $1/\ln(x)$.

- Without using Graphing Calculator, can you tell where

$$y = \frac{2}{(x-7)(x+2)}$$

explodes? Verify that using the software.

- Write two different functions that explode at both $x = -1$ and at $x = +2$.
- Write a function that has a zero at $x = 1$ and explodes at $x = 3$.
- Some of the following graphs are reciprocals of others. Find the reciprocal pairs and explain how you know they're reciprocals. Then see if you can come up with equations that produce graphs like the ones shown.



- Design a 3D surface that explodes along the circle $r = 1$.
- Design a 3D surface that explodes at $(1, 1, 8)$ and nowhere else.

Inverse functions

A function is a rule for changing x to y . The inverse of a function is how you get back from y to x . We're exchanging x and y , so the

graph looks like we took the Cartesian plane and flipped each axis into the other. For example, if my function is $y = x^2$, the inverse is $y = \sqrt{x}$.

This is a good example because not all functions look exactly like their flipped inverses. In this case, in order for the inverse to be a *function*, that is, to give at most one value of y for each value of x , the curve loses its other half. What is the square root of 4? The function says 2, but we know that -2 is another answer.

How do you get Graphing Calculator to help you find inverses?

- ◆ Write the function. We'll use the exponential $y = 2^x$.
- ◆ Then select every y in the equation ...

$$y = 2^x$$

- ◆ ...and change it to an x .

$$x = 2^x$$

- ◆ Then select every (original) x and change it to a y .

$$x = 2^y$$

- ◆ Finally, select the y and choose **Isolate Term** from the **Math** menu.

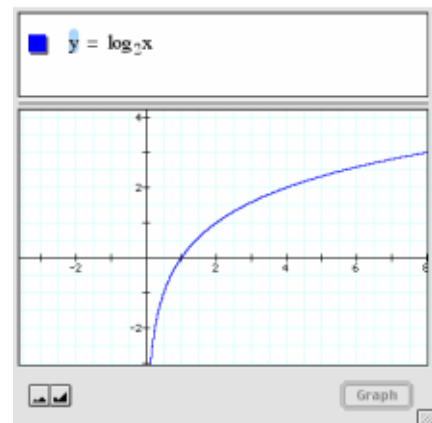
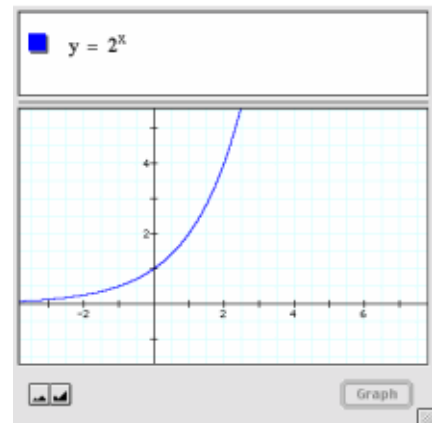
$$y = \log_2 x$$

Aha! Logarithms are the inverses of exponentials, and indeed the curve looks the same, just flipped and sideways—as if you had exchanged the x and y axes.

Things to do

Use Graphing Calculator to graph the following functions and their inverses. If the graphs do not create perfect mirror reflections of one another—e.g., $f(x) = x^2$ and its inverse $f^{-1}(x) = \sqrt{x}$ —explain why not.

- ◆ $f(x) = -3x + 2$



◆ $f(x) = e^{-2x}$

◆ $f(x) = \frac{1}{x}$

◆ $f(x) = \tan x$

◆ $f(x) = x^n$

◆ $f(x) = \sqrt{1+x^2}$

◆ $f(x) = \sinh(x)$

This function is called the *hyperbolic sine*. You may never have seen it before—but you can graph it. Type it in or use the keypad.

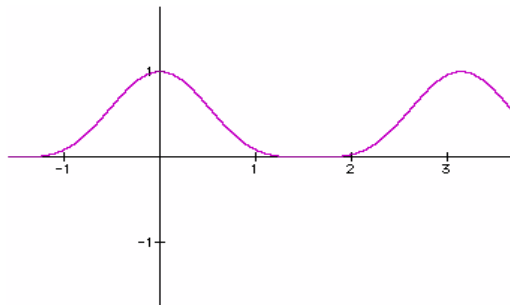
◆ There are some functions with inverses, e.g., $f(x) = \ln x + x$, for which you can't use that straightforward Graphing Calculator technique of substituting and isolating. What about that function makes it impossible to use that technique?

◆ If you had two functions alleged to be inverses of one another, how could you use Graphing Calculator to see if it was true?

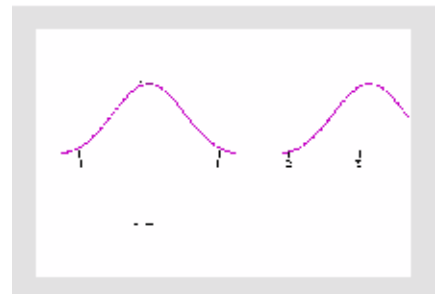
Try your idea on the pair of functions $f(x) = x^2 - 4x + 6$ and $g(x) = 2 + \sqrt{x-2}$.

Functions of functions: composition

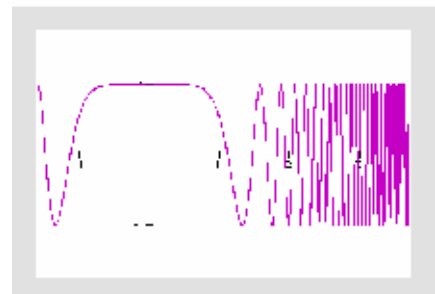
We've translated functions, dilated them, taken their reciprocals, found their inverses, added them, and multiplied them. One more technique: we can *compose* them, that is, we can take functions of functions.



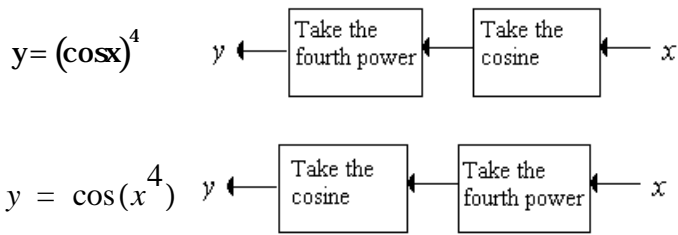
We'll look at two examples at first, using the same two functions. The two functions are $f(x) = \cos x$ and $f(x) = x^4$. Notice that order matters when we compose them. There are two ways:



$y = (\cos x)^4$ aka $\cos^4 x$



$y = \cos(x^4)$



What makes the graphs look so different? In the top example, once we take the cosine, all we can get are numbers between -1 and $+1$. So when we take the fourth power, we get numbers between 0 and 1 , a periodic function with twice the frequency of the original cosine (because all of the “down” humps got converted to “up,” doubling the number of humps). In the bottom example, we take the fourth power first, so that when we take the cosine (yielding numbers between -1 and $+1$, as before) it’s as if x is changing very quickly as we move away from the origin.

It’s not always obvious what we’re going to get. But some principles help us get the functions we want through composition, based on features of functions we see in the world. We’ll discuss how to control oscillation, explosions, and how functions go to zero.

Here are two lessons about oscillation from the example:

- ◆ If you apply a sinusoid first and then another function, you’ll get an oscillating function where each period is shaped according to the way the second function deals with numbers between -1 and $+1$.
- ◆ If you apply a sinusoid second, you’ll get a function constrained to be between -1 and $+1$, but whose “frequency” changes depending on the first function.

A traditional notation for composing two functions is $f(g(x))$. That is, you apply g first, and then f .

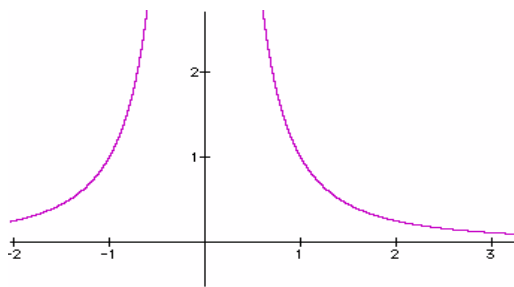
We’ve already discussed explosions; when you take the reciprocal of a function $g(x)$, you’re doing a composition by applying $f(x) = 1/x$ to the results of the function. So here’s another lesson:

If you apply $f(x) = 1/x$ second, all of the zeros of the first function will become points of explosion (e.g., $1/\sin(x)$ explodes at $x = \delta$).

That helps us with explosions, but what about constructing composite functions that go to zero? One way is to use functions we know that already go to zero such as $f(x) = 1/x$ and $f(x) = e^{-x}$ posing functions, we can put anything in place of the x . The only requirement is that the first function, $g(x)$, grow without bound.

If you use a function $g(x)$ that grows without bound first and then apply $f(x) = 1/x$ or $f(x) = e^{-x}$ (or any other function that goes to zero as $x \rightarrow \infty$), the composite function $f(g(x))$ will go to zero wherever $g(x)$ grows without bound.

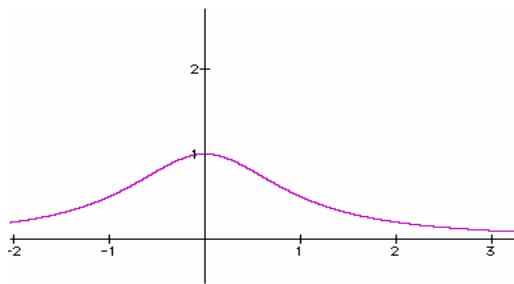
Making a Hump



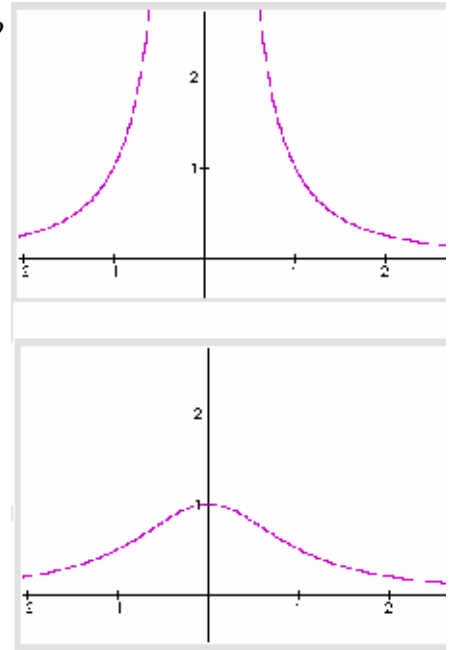
Here's an example: suppose we want a function that is a hump. It's going to have a height of one, the hump will be centered on $x = 0$, and it will trail away to $y = 0$ as $x \rightarrow \pm\infty$. What should we do?

Strategy 1: Let's see if we can use $f(x) = 1/x$. We need something to go into the x place that grows without bound both as $x \rightarrow \infty$ and $x \rightarrow -\infty$. Let's try $g(x) = x^2$. Then we'd get our composite function,

$$f(g(x)) = \frac{1}{x^2} \text{ (shown in the margin)}$$



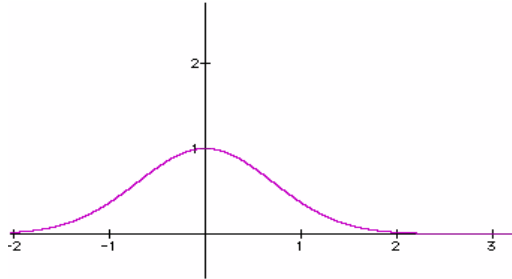
It explodes at $x = 0$, and we want it to be one. The problem is the denominator goes to zero; if it were only one, the quotient would be



one and we'd be set. Let's add one to the denominator, to get

$$f(g(x)) = \frac{1}{x^2 + 1}$$

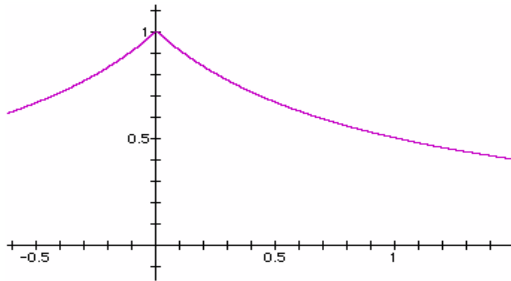
which is just what we had in mind.



Strategy 2: Start with $f(x) = e^{-x}$. Again we want a function to stick into x that will grow without bound as $x \rightarrow \infty$ and $x \rightarrow -\infty$. Again, we try $g(x) = x^2$. Our composite function is $f(g(x)) = e^{-x^2}$ —the function in the margin.

This function is famous. Its shape is called a Gaussian—it's also called the “normal” curve.

Things to do

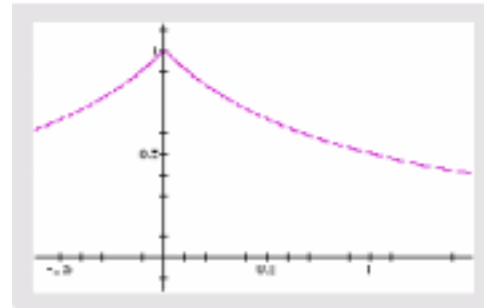


These problems use all sorts of transformations as well as composition.

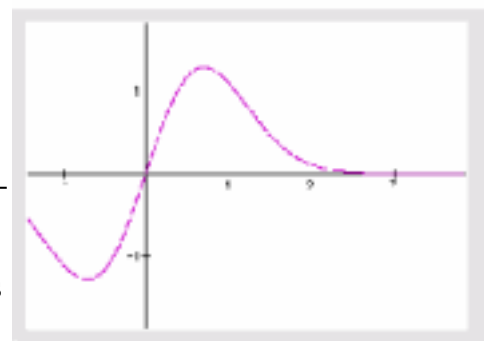
- ◆ Design another hump function that goes to zero at ± 8 and has its maximum at $(0, 1)$.
- ◆ Find the equation for the pointy hump in the margin.
- ◆ Use the parameter n and the Gaussian function $y = e^{-x^2}$ to make a graph which, when animated, shows a hump sliding across the screen.



Gaussian

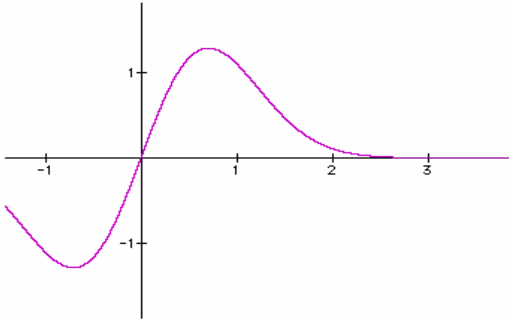


A pointy hump



A mountain reflected in a lake

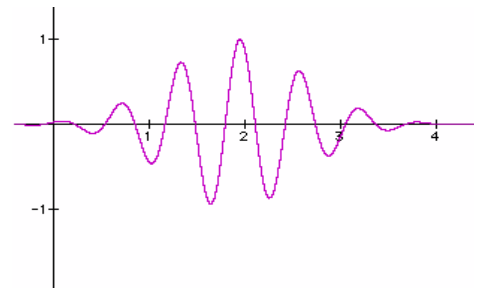
- ◆ Use n and $y = e^{-x^2}$ to make a graph which, when animated, shows a hump getting taller and shorter without changing position.



- ◆ Use n and $y = e^{-x^2}$ to make a graph which, when animated, shows a hump getting thicker and thinner without changing height.
- ◆ Design a function with one up and one down hump, such as (but not necessarily identical to) the one in the margin.
- ◆ Design (or discover) a function that changes smoothly and monotonically from 0 at -8 to 1 at $+8$. Depending on your approach, you may want to think of a ratio of two functions where the numerator goes to zero as $x \rightarrow -\infty$; two functions approach each other as $x \rightarrow +\infty$.
- ◆ Change the function from the previous problem so that its transition from 0 to 1 is as sharp as possible and happens at $x = 0$.

Hint: Compress it horizontally.

- ◆ Design a function that (as far as possible) has a value of 1 between -1 and $+1$ and zero everywhere else. Don't use $\text{sgn}(x)$.
- ◆ Make a wave packet—a function with a hump as an envelope like the one in the margin. Animate it so it moves across the screen.
- ◆ Another composition rule appears below. Explain what it means and why it works, and give examples of where you might use it.



If the function $f(x)$ has the form $f(x) = x^n$, and n is even and positive, then $f(g(x))$ will be positive. Those parts where $|g(x) < 1|$ will tend to zero the larger n gets, and where $|g(x) > 1|$, $f(g(x))$ will grow.

Surfaces in 3D

One of the best things about Graphing Calculator is its ability to graph surfaces. These functions are like landscapes. Imagine that the (x, y) plane is horizontal. The x -axis runs east and west, the y -axis north and south. A function of x and y is a rule that tells you the height of the landscape.

These functions often have both x and y in them, though they don't have to.

Basic 3D drawing in Graphing Calculator

Many 3D commands and techniques are the same as 2D—you just have more options.

- ◆ Start with a fresh window.
- ◆ In the math pane, enter $z = xy$ and graph the function.

You'll see a pointy, saddle-shaped hyperboloid.

- ◆ If it is rotating, click it to stop it.

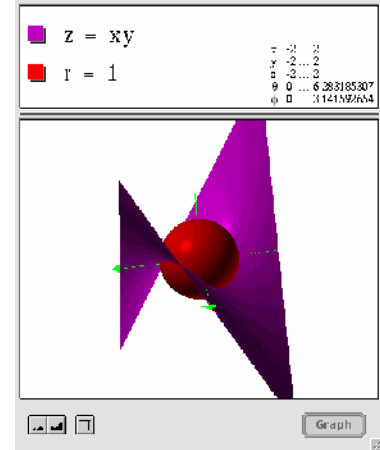
Click, drag, and release, in any direction, to start it rotating again. The speed of your release controls the speed of rotation. Notice that the surface is lit correctly so you can see its curves; also, notice how rotation helps you see the shape.

But rotation can also get you disoriented.

- ◆ Choose **Reset Graph** from the **Graph** menu to reset the axes, orientation, and rotation of the figure.

Let's add another surface!

- ◆ Choose **New Math Expression** from the **Math** menu to get a new spot in the math pane.
- ◆ Enter the expression $r = 1$ and graph what you have.
You've added a sphere to your graph. It should look something like the picture at right.
- ◆ Click the colored boxes next to the equations to change the colors and patterns on your surfaces.
- ◆ By default, the figure rotates around the z -axis. You can make arbitrary rotations by dragging.
- ◆ Rotate the figure about the x -axis by pressing **⌘-option-x** (on the Macintosh) or **ctrl-shift-X** (on Windows). Rotate the figure about the y -axis or z -axis by pressing **⌘-option-y** or **ctrl-shift-Y** and **⌘-option-z** or **ctrl-shift-Z**.

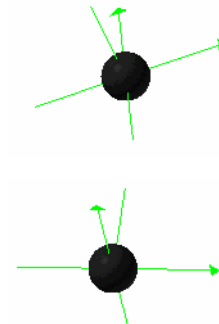


How to tell which axis is which

This is a small thing, but if nobody tells you, how will you ever know? In Graphing Calculator's 3D world, the x - and y -axes have arrows that point in the positive direction. The z -axis has no arrow, but Graphing Calculator only draws its positive side.

It gets tricky when you have spun the coordinate system, especially when the z -axis points away from you. You may have to rotate the set of axes mentally so that they're in some sort of standard configuration. Or you may need to remember—or learn—a right-hand rule: if the thumb of your right hand points along the z -axis, your fingers curl so that they get to the x -axis just before they get to the y -axis.

In the top illustration at right, x goes right, y goes into the page and up, and z goes up. In the bottom illustration, x goes *up*, y goes *right*, and z goes in and up.



Cool ways to slow down your computer

It is a truth universally acknowledged that any computer possessed of a fast processor must be in search of something that will bring it to its knees. And Graphing Calculator can do it. It's not the program's fault, unless you blame it for giving us the opportunity to try

outrageous things. Some of them are really wonderful, too, so they're worth the degradation in performance.

First, there is a new control below the graph pane. It controls the *resolution* of the surfaces. Left is coarse, right is smooth. Of course, that means that left is also faster and, depending on how complex your function is and the speed of your processor, right can become really slow. Try “coarse” first, and see what happens to the sphere.



Second, for many fast computers, there is an item in the **Graph** menu called **Use Transparency**. Put your cursor in an expression in the math pane and choose the item. *That surface* becomes a little transparent.

Transparency looks great, and it may help you visualize what you have on your screen, but it will cost you in speed of rendering and rotation.

All that said, let's give the program a break! Think of what it's doing. It breaks up each surface into a jillion polygons (depending on resolution), computes their vertices, and determines where they project onto your plane of vision—in perspective. Then, for each polygon, Graphing Calculator computes the angle between its perpendicular and the light source to see how to shade it. If there's more than one surface, Graphing Calculator does the same for the other surface *and has to tell whether each polygon is in front of the other surface or not*. If you use transparency, Graphing Calculator shades the “see-through” area still differently! Then, if you're rotating, it has to do that whole thing at least ten times a second or you'll complain that it's jerky. Sheesh! What an ingrate you are! Why don't *you* try it, huh?

Things to Notice

Play around! As you do, here are three things to notice:

- First, notice that we can graph expressions from two different coordinate systems—Cartesian and spherical-polar—simultaneously. And we can graph as many of them as we like.

- While dragging moves axes or traces functions in 2D, it controls rotation in 3D.
- The range of coordinates or parameters appears at the right edge of the math pane. You can edit those values to scale and recenter your graph. For example, when you make the sphere, you can see in the math pane. Edit that 6.28 to be a smaller value, say, 2.0. What happens to the sphere? (You'll need to press **enter** to get the graph to reappear.)

Things to do

Here are some things to try that will teach you how Graphing Calculator handles three-dimensional surfaces:

$z = nxy$	Yes, n works.
$z = x + y$	Adding and subtracting x 's and y 's will give you planes tilted one way or another.
$z = x\sqrt{y}$	When part of the graph is undefined, Graphing Calculator just leaves it empty.
$z = 2 - (x^2 + y^2)$	This is called a <i>paraboloid</i> —a 3D parabola.
$z = \sin y$	You can also add a coefficient to the y . See how, since x doesn't appear, the function is the same for all values of x .
$z = \sin(x^2 + y^2)$	Here we have a sine again, but this time it depends on the square of the distance from the origin.

A Family of Graphs at a Glance

One reason people use 3D plots is to show how changing a parameter changes a function. We've talked about how you can investigate changing parameters with n , but 3D plots let you create static pictures of many curves at once. It's as if each slice of the 3D surface is a curve with a different parameter. How do you do it? Suppose we want to see how x^2 changes with its coefficient. You could use

$$y = nx^2$$

but that will only show you one curve at a time.

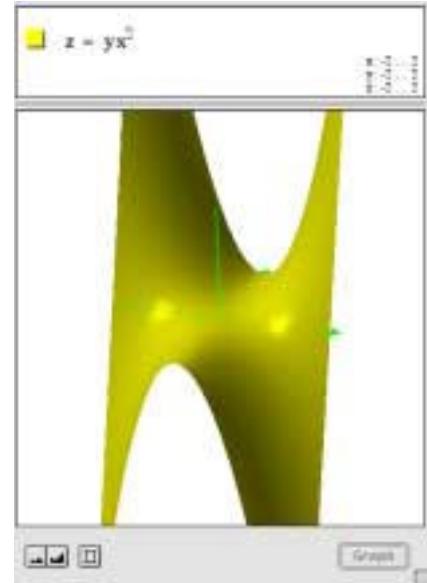
Instead, let's make a surface with

$$z = yx^2$$

You'll get the screen in the margin, where the x -axis points right, and y points into the page. You can see the parabolic shape of each "slice" going from left to right. In the middle it's flat. Since $y = 0$ there, the function is flat as well. Close to us, the parabolas point down since y is negative.

To make a surface that shows the effect of a parameter, replace the parameter with y and set the resulting function equal to z .

Here are some functions you should plot to see how this works:



The basic surface Try this first	Then use a parameter, y	Comments
$z = \sin x$	$z = y \sin x$	What is the effect of multiplying $\sin x$ by a parameter? It stretches, right? This surface gets clearer if you zoom out to see more wiggles.
$z = \sin x$	$z = \sin yx$	What if you put the parameter y next to the x ? It squishes the curve horizontally, showing more wiggles the bigger y gets. So the surface is flat on the x axis and wigglier the farther you get from it. Zooming out helps here, too.
$z = x^2 - 1$	$z = (x - y)^2 - 1$	This shows how the y makes the parabola slide back and forth. Notice how the axes stick through the surface.

Things to do

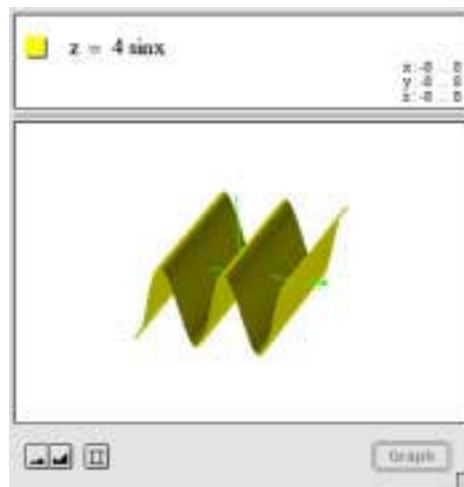
For one of the rows in the table above, comment on what's happening, perhaps as we did above for $y = x^2$. One way to approach this is

to look at the original function in two dimensions with the parameter n instead of y , and let n take values between -2 and $+2$.

Surfaces as Genuine Functions of Two Variables

People don't use $z = f(x, y)$ only to show the effect of parameters. Sometimes you have two independent variables, and a third quantity that depends on them. Maybe x is the temperature, y is the price of lemonade, and z is the sales you get that day at your lemonade business. Sales go up with x and down with y .

Sometimes, though, the surface is the value of some variable over a domain that's actually an x - y plane—for example, some waves in water—and the surface is the surface itself. In the margin you see some waves going east-west. We could use $z = \sin(y)$ to make them go north-south.



Things to do

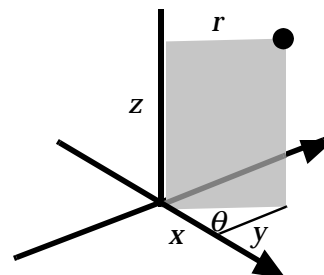
- ❖ How would you get the waves to go in some other direction—southwest, say?
 Could you do it without affecting the amplitude or period of the waves? How would you investigate that?
- ❖ What happens when two wave patterns overlap?
- ❖ Make a 3D surface that gets higher as you get farther from the origin in all directions.
- ❖ Can you make an animation (using the parameter n) that shows waves moving along?

Suggestion: Do it in two dimensions first.

Surfaces of revolution

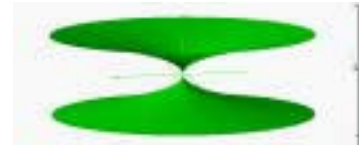
Up above, where we mentioned cylindrical coordinates, you might have thought, “where am I going to use cylindrical coordinates?” Let's look at two examples: first, centrally, in making surfaces of revolution; second, incidentally, in making conic sections.

Before we start, let's figure out what cylindrical coordinates are. They're just like polar coordinates in the plane, except you have a z -axis sticking up out of the origin. So you can represent every point in 3D-space with an ordered triple: (r, θ, z) . The r -coordinate is the distance of the point from the z -axis, radially. The z is the dis-



tance above the x - y plane. And theta, as in regular polar coordinates, is the “azimuthal” angle of the point: if you project the point onto the x - y plane and shoot a ray through that projection from the origin, theta is the angle the ray makes with the positive x -axis, measured counterclockwise. The picture says it better.

A surface of revolution is what you get when you take a two-dimensional curve and spin it about an axis. Typically, you spin it about the x -axis. So if you have $y = x^2$, say, and spin it, you get something that looks remotely like a trumpet heading off in both directions.



How do we do the spin? The easy way is to rewrite the function so that it is r as a function of z instead of y as a function of x . That is, instead of $y = x^2$, write $r = z^2$. You’ll see something like this:

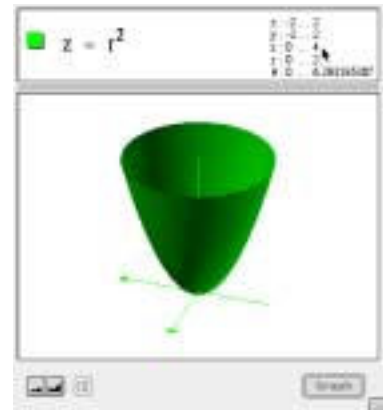


The whole thing is on its end, of course—by default, z points up, while x points to the right—but that really doesn’t matter. If you rotate it back onto its side, you’ll see the parabola you expect:

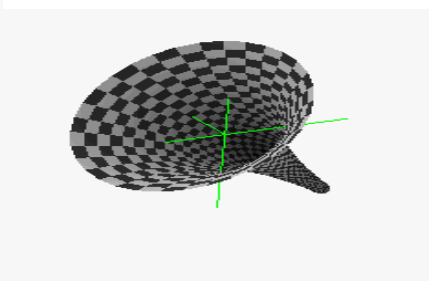
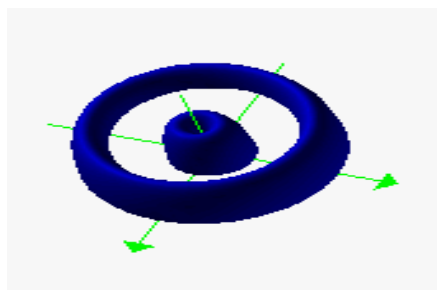
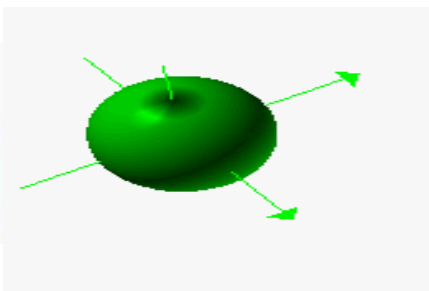
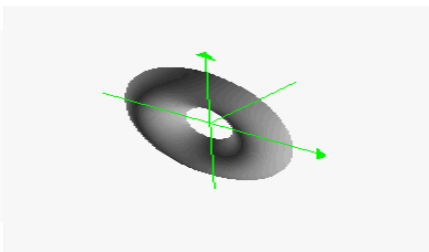
To rotate around the y -axis instead, just reverse the r and the z :

Things to do

- ❖ Explain clearly why $r = z^2$ is the same as $y = x^2$ rotated around the x -axis, while $z = r^2$ is the same as $y = x^2$ rotated around the y -axis.
- ❖ Try rotating various functions around the x - or y -axes to see how they look. (e.g., $z = \sin(10r)$)
- ❖ Add sliders to your functions. For example, try $r = z^2 - n$; let n run from -2 to 2 in 40 steps.



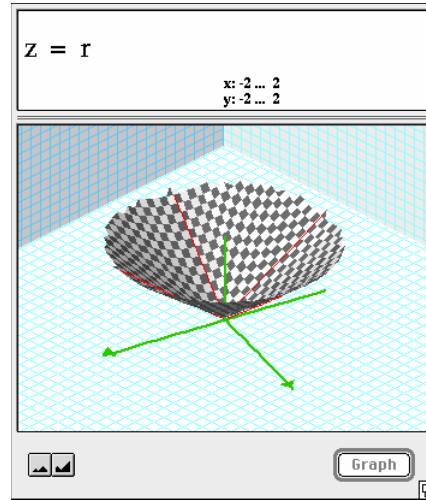
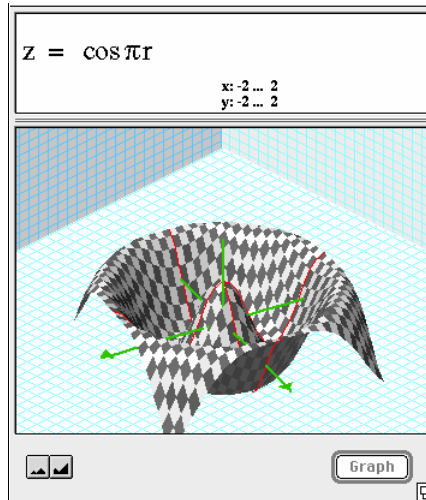
◆ See if you can make these (Solutions on page 146):



Polar Coordinates and Surfaces

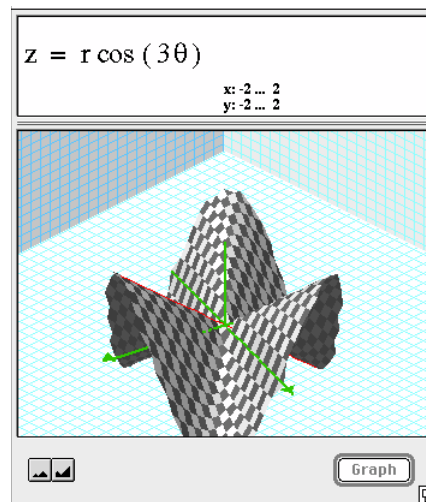
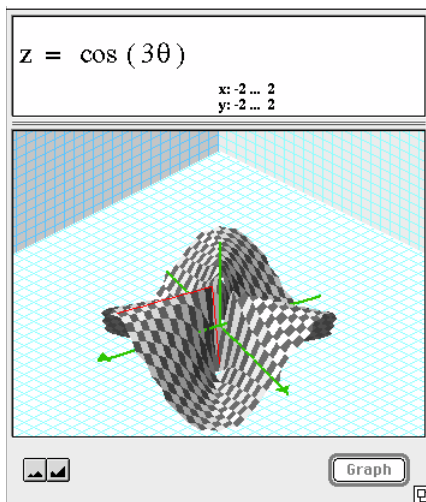
Since r and θ are coordinates on a plane, we can write functions of them when we make surfaces. You can combine x , y , r , and θ any way you want in a function z . Some examples follow.

This function undulates as it gets farther from the origin. Since it's a function only of r , the waves are circular. The π is there to make the first maximum at $r = 2$.



Here, the height of the surface is equal to the distance from the origin. This makes a cone.

Because of the three, this function oscillates three times as it proceeds around the circle. The landscape is steep near the origin because different values of θ can be so close together.

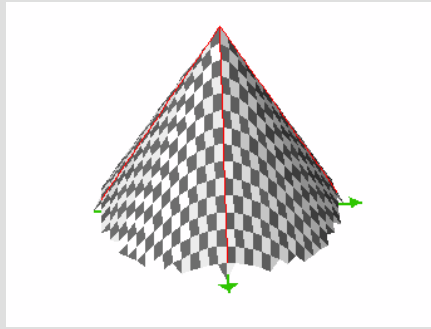


This is a combination of the function above and the one to the left; see how the amplitude of the oscillations goes to zero as r goes to zero.

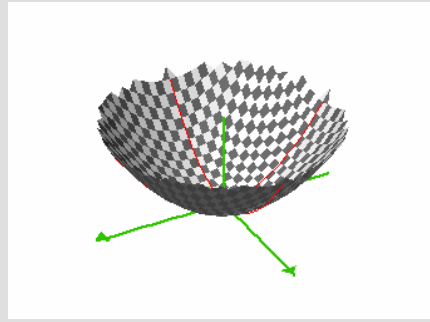
Things to do

See if you can come up with equations for surfaces like these:

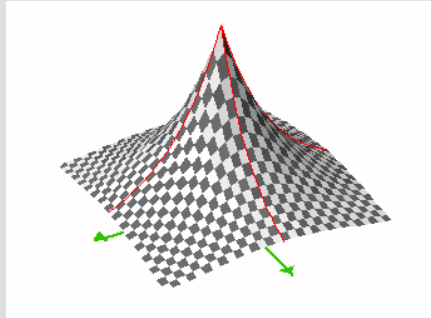
1.
This is a cone,
but this time its
point is up
instead of down.



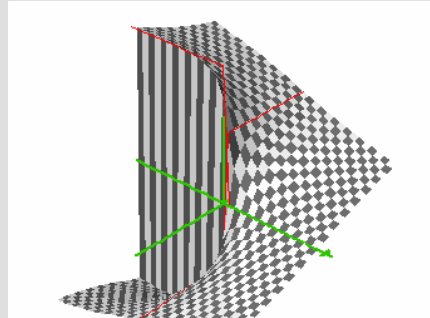
2.
Think “parabola.”



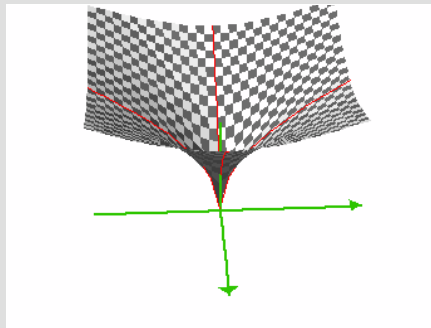
3.
Think “expo-
nential decay.”



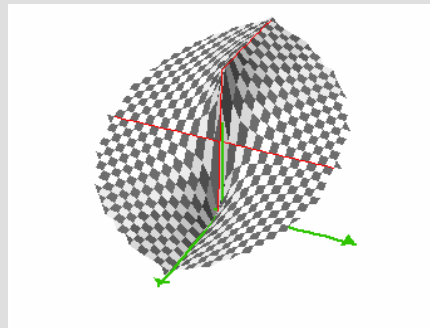
4.
Looks hairy, but
it’s not. It is zero
along the posi-
tive x -axis
(coming out of
the page to the
right) and
discontinuous at
the negative x -
axis, where it’s
equal to π on
one side and $-\pi$
on the other.



5.
You can get
something like
this by thinking
“parabola” or
“upside-down
exponential
decay.”

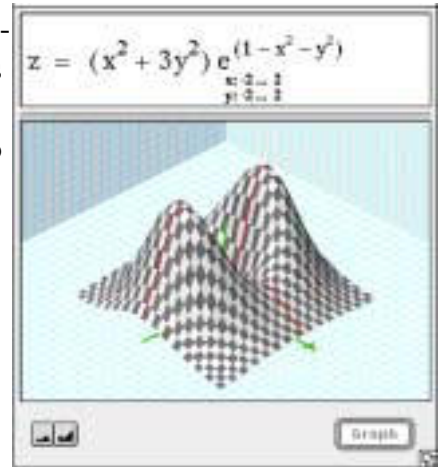


6.
In this graph—
which is a lot
like the one
above—the
function has its
minimum along
the positive x -
axis.



Slicing 3D Surfaces

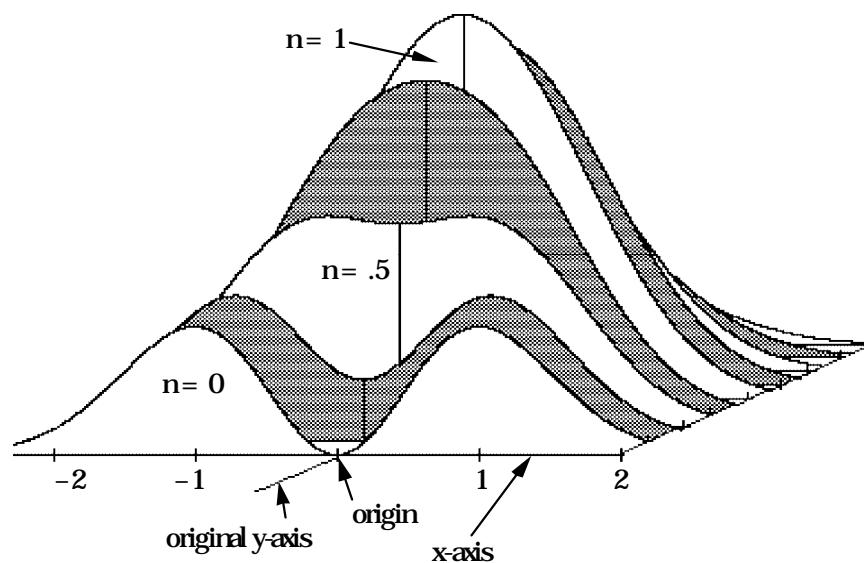
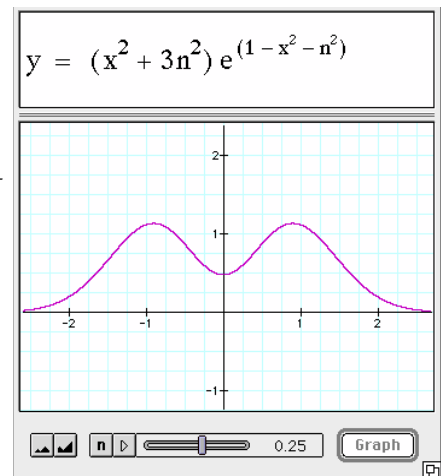
You can use Graphing Calculator to slice up 3D surfaces both vertically and horizontally. Let's look at vertical slices first. We'll use as our example the "two hills" function shown in the margin. To slice vertically, think of the surface as a bunch of functions pasted next to each other, each slightly different from the next. They're the parallel lines on the checkerboard that make the surface in the graph window. To get those functions to show up as two-dimensional functions, you have to do some substitutions.



Suppose you want to slice the surface parallel to the x -axis. Then we'll have to call the old z -axis y in order to get Graphing Calculator to graph it. What happens to y , then? You choose the value of y at which you want to slice the surface. If you don't want to choose, use n , and you can choose later! So do this:

- ◆ Substitute n everywhere you see y .
- ◆ Substitute y everywhere you see z .
- ◆ Graph it!

Adjust n to an appropriate range (see page 10; here an appropriate range is -2 to 2). If you animate the graph, you'll see a movie where each frame is a slice of the surface at a different value of n —that is, where the slice cuts the original surface at a different value of y .



It's more complicated to slice parallel to the y -axis, because then y becomes the horizontal axis in the two-dimensional graph, which Graphing Calculator wants to be x . The former x must become the slicing position, n . So you have to change x to n , y to x , and z to y .

Suppose you want a *horizontal* slice of the same “two-hills” function. Think of the surface as a landscape and the horizontal slice as a sea level that raises or lowers. Use an inequality (the slice will not necessarily be a function). Get rid of the equals sign and the z , and instead use n and either $>$ or $<$. The parameter n becomes the sea level. In the graph in the margin, the shaded area is the ocean.

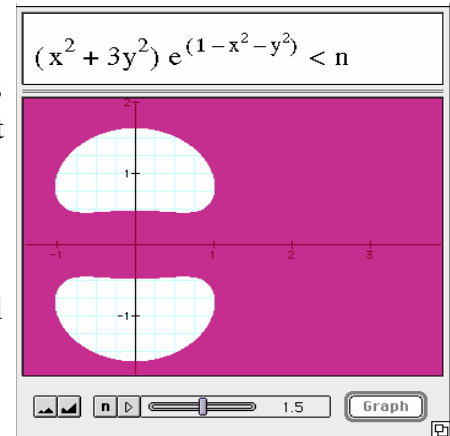
If your computer is fast enough, you can animate this graph; you'll see the islands growing and shrinking as the sea level rises and falls. Older Macintoshes may be too slow.

Note: If your surface has r or θ in it, this technique will not work! You have to recast it as a function in x and y . Remember that

$$r = \sqrt{x^2 + y^2}$$

Things to do

- ◆ Here we used $f(x, y) < n$, and we got the shaded part to be the ocean. Explain why that worked, that is, why was it “less than n ” instead of “greater than?” Is that true for all functions $f(x, y)$?
- ◆ Use Graphing Calculator and this sea-level technique to find the maximum and minimum values for this two-hills function where x and y are both between -2 and $+2$.
- ◆ The sea-level inequality looks strange for $n = 1$. That means something strange is true in the surface at $z = 1$. Explain it.
- ◆ Why does the two-hills function, $z(x, y) = (x^2 + 3y^2)e^{(1-x^2-y^2)}$ have two hills?

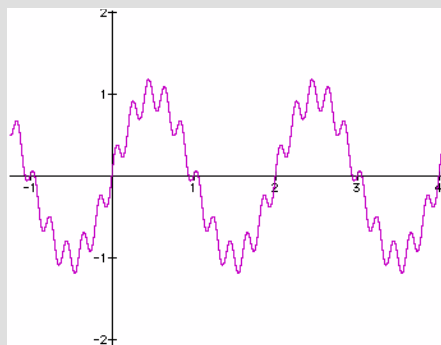


Challenges

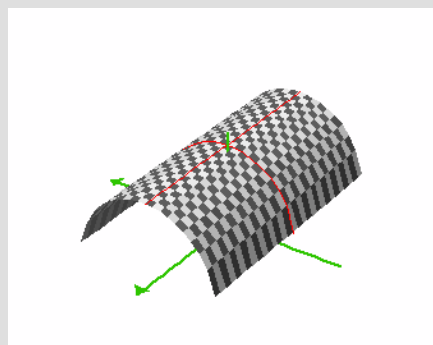
Here are some strange graphs. See if you can make them!

<p>1. Yikes! Stripes!</p>		<p>2. Frequency modulation</p>
<p>3. A stump</p>		<p>4. Rotated parabola</p>
<p>5. Napkin</p>		<p>6. The letter W</p>
<p>7. Graphing a bullseye</p>		<p>8. Solid diamond</p>

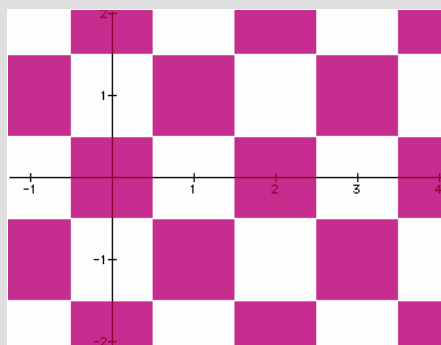
9.
Multiwiggle



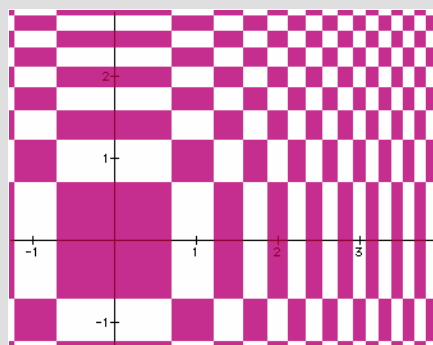
10.
Quonset hut



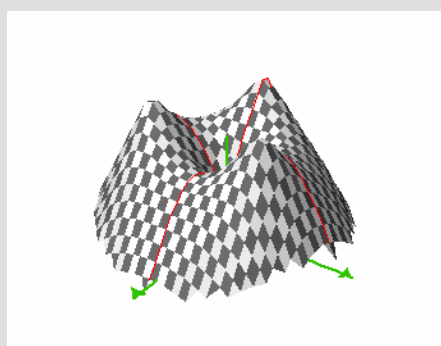
11.
Check!



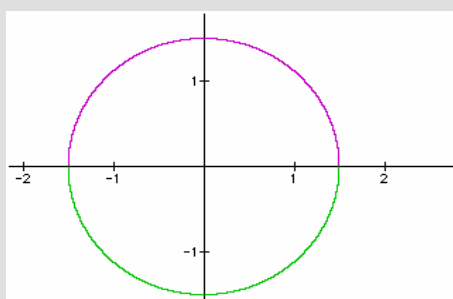
12.
Op art



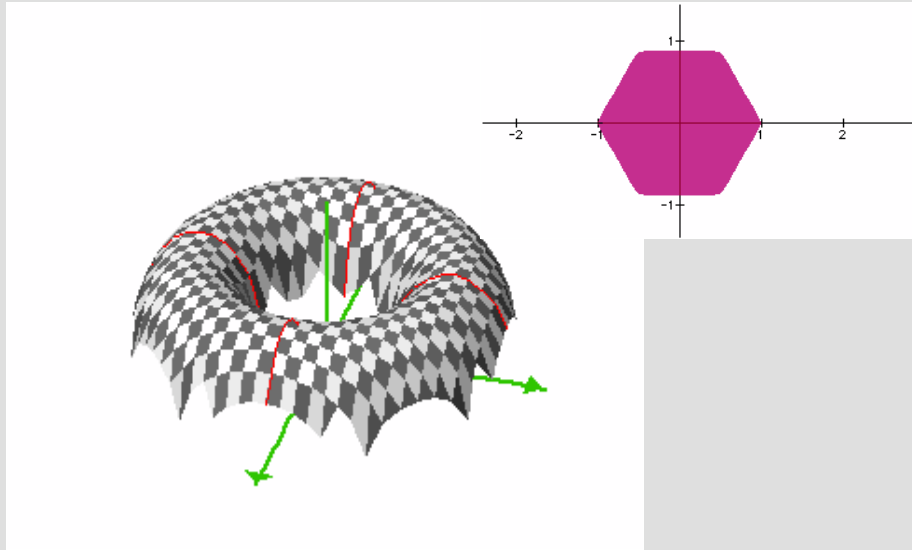
13.
Three peaks



14.
Graphing a
circle—not
its interior

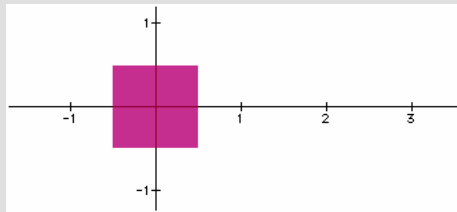


15.
 Floating donut

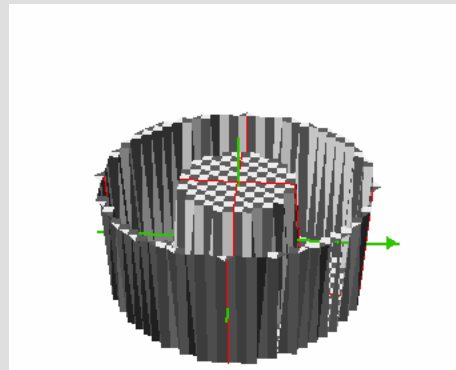


16.
 Hexagon

17.
 Solid square



18.
 Bundt pan



19.
 One of your
 own

20.
 Another one

Conic sections—using cones!

3D surfaces can help you illuminate mathematical ideas. If you have a commercial version of Graphing Calculator you can look at the intersections between surfaces. (If you have version 1.3, see “Inequalities” on page 26.)

We’ll look at some of the more famous intersections: conic sections that result in a circle, ellipse, parabola, and hyperbola. Unlike most treatments, however, we’re actually going to slice up a cone to get them.

Graphing Calculator can handle a wide variety of implicit functions, including the traditional conics. For example, you can easily plot these (and you should try them, it's good practice):

$$x^2 + y^2 = 2 \qquad \frac{x^2}{2} - \frac{y^2}{3} = 1 \qquad x + 4y^2 = 0$$

and maybe the coolest of them all, $x^2 = y^2$

You can also include parameters, for example:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Make any of the constants into a variable slider so you can see its effect on the curve. You can only make one slider at a time.

Then there are some pretty wild implicit functions that are not conics. Play around and see what you get. Here is an odd one:

$$\cos x + \frac{y}{x} = \sin y + \frac{x}{y}$$

You'll have to zoom out to get the full effect. The odd picture you'll see in Graphing Calculator's graph pane—which I bet you've never seen before—brings up an important principle of learning about graphs using technology:

With a program like Graphing Calculator, you can experiment in ways you have never experimented before. And even if you're really good at graphing, Graphing Calculator will quickly exhaust your intuition. This is good: it gives us a chance to develop new intuitions, and to explore familiar math in new ways.

◆ Start with a blank Graphing Calculator document.

First we make a cone that has no base—it extends forever. Here is one way to make it.

◆ Enter $\mathbf{z} = \mathbf{r}$ and graph it.

You'll see the tip of an infinite cone, but we need a cone that extends in *both* directions—two cones touching at the tips.

◆ Here are three ways to do this:

- Make a second cone using $z = -r$.
- Change your $z = r$ to $r = z$ —interesting, huh?
- Select the z in the math pane and type a vertical bar (it's probably above the **enter** or **return** key).

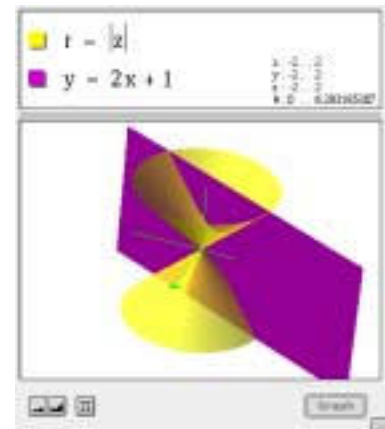
That makes the expression $|z| = r$. That gives us what we want.

Take a moment to understand how each formula for a cone works. Because we're using z and r in the same equation, Graphing Calculator thinks we're using cylindrical coordinates for this surface. That is, the height above the x - y plane is the same as the horizontal distance radially from the z -axis.

We could also have written $z^2 = x^2 + y^2$ to get the same result.

Now let's proceed. We want to slice this thing with a plane. We'll start with a simple one.

- ❖ Choose **New Math Expression** from the **Math** menu.
- ❖ Enter $y = 2x + 1$.
- ❖ Graph it to see a plane cutting through the cones, parallel to their axis.



Look at the intersection of the plane with the cone. This is a hyperbola. Like the familiar 2D hyperbolas, it has two parts that open away from one another. (In this illustration, the cones have transparency, but the plane does not. If you have a fast processor, try it!)

The formula makes a plane instead of a line because Graphing Calculator knows we are graphing in three dimensions. Look carefully at the plane. Point the z -axis at yourself and rotate it so that one axis-arrow is pointing right and the other is pointing up. The plane is the same as the familiar line, swept through the z -axis.

Any expression that contains only x and y will be parallel to the z axis, so we'll only get hyperbolas.

- ❖ Replace that equation for the plane with each of the following (don't just add them or it will get too cluttered).
- ❖ For each one, try to figure out (a) why the plane is where it is and (b) what shape the intersection makes:

$$z = 1$$

$$z = \frac{1}{2}x + \frac{1}{2}$$

$$x + y + z = \frac{1}{2}$$

$$z = y + 1$$

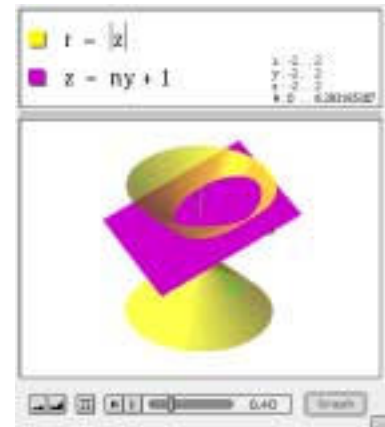
Let's make a figure where you can control the slope of that plane so that, without retyping, you can make all the different conic sections. Just as with 2D functions, we can use a slider.

- ◆ Add an n into the last expression as a coefficient for y . That is, it now reads $z = ny + 1$.
- ◆ Change the slider range so the **Lowest Value** is 0 and the **Highest Value** is 2. Give it 40 steps.

Note: Display the dialog box by choosing **Slider Values** from the **Graph** menu or by clicking the **n** in the slider control itself.)

- ◆ Play with the slider and see what happens.
- ◆ Rotate the figure by hand to see the most interesting parts.
- ◆ See if you can figure out which values for the slope n give which conic sections and why.

It's interesting to zoom out to see if the plane hits the cone farther out. When you do, note that the hyperbolas get "pointy" and the parabolas close up. Why is that?



More general 3D surfaces

There is a lot more to surfaces in Graphing Calculator than surfaces of revolution. Once you leave the land of $r = f(z)$ or $z = g(r)$ in cylindrical coordinates, most anything can happen.

In the original Graphing Calculator, you can make z any function of x and y , that is, $z = f(x, y)$. This is what we did first with $z = xy$. For any pair of numbers (x, y) there is one value of z . It's like a landscape.

Now, however, you can use x , y , and z as Cartesian coordinates in any combinations, making implicit functions. Try these:

$$xyz = \frac{x+y+z}{4} \quad \cos x + \cos y + \cos z = 1 \quad z^2 = 2^y \cos(\pi x)$$

You can also use θ (longitude—in radians—for cylindrical and spherical coordinates) or φ (latitude in spherical) in similar ways. To enter θ , type “theta”; for φ , type “phi.” Try these—and see if you can figure out how they work...

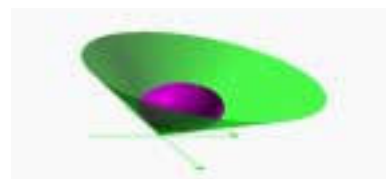
$$r = \theta + \varphi \quad r\theta = \varphi \quad \theta = \pi z$$

Note: There are many possible combinations that Graphing Calculator can’t graph, such as $r + z + \theta = 1$. But it can often graph equivalent equations such as $r = (1 - z) - \theta$.

Note: You can graph surfaces using different coordinate systems on the same graph, just as when we plotted $z = xy$ (Cartesian) with $r = 1$ (spherical).

Things to do

- ◆ Make the figure at right. The sphere is tangent to the cone
- ◆ Next, control it with a slider. Make it so the number in the slider is the z -coordinate of the center of the sphere.



Exploring Parametric Equations

You usually don't study parametric equations until you take advanced algebra or pre-calculus. Even then, you may not spend much time with them because they can be hard to visualize. The commercial version of Graphing Calculator makes it easier—and gives you the chance to make some pretty wild things. We'll start in two dimensions and move on to three pretty quickly. We'll also move on from parametric curves to parametric surfaces.

Note: Graphing Calculator 1.3 doesn't support 2-vectors or 3-vectors, so the material in this chapter won't work in that version.

Remind me about parametric equations

In two Cartesian dimensions, you have two equations—one for x and one for y . These are both functions of the parameter, which we'll call t . So we might have equations like these two:

$$x = \cos t$$

$$y = \sin t$$

We start with $t = 0$ and calculate x and y . Those two values are $\cos(0) = 1$ and $\sin(0) = 0$, respectively. So $(1, 0)$ is one of the points in our curve. Now, as t increases, we keep calculating x and y , plotting the points (x, y) . That's the basic idea.

Note: You have to define the range of t . In this case, we'd probably choose $[0, 2\pi]$ since our trig functions are in radians.

Making parametric curves in Graphing Calculator

How do we get two equations attached to the same curve? We have to use vectors.

❖ Start with a new Graphing Calculator document.

❖ Choose **2-Vector** from the **Math** menu.

You'll see a 2-vector appear. The shortcut for **2-Vector** is **⌘-2** (on the Macintosh) or **ctrl-2** (on Windows).

❖ Type an equals sign.

❖ Once again choose **2-Vector** from the **Math** menu.

You will see something more or less like this:

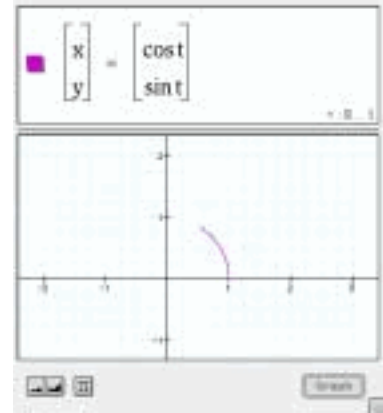
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

- ◆ Change the right-hand side so that the parametric expressions correspond to the x and y, like this:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

- ◆ Now press **enter**.

Your window should look something like the one at right.



This shows the curve, but for what range of t ? You can find the answer on the right side of the math pane, just above the split bar. Graphing Calculator is using the default range, $0 \dots 1$. That is, the curve goes from zero to one radian. We need it to go to 2π .

- ◆ Edit the equation to be

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{bmatrix}$$

- ◆ Press **enter** again and you'll see the whole circle.

Note: We could also have fixed the circle by editing the range of t to be $0 \dots 6.28$, but the way we did it is better for things we'll do later.

Things to do

Edit the expressions in the *right-hand* vector, changing them in order to make transformations—translations and dilations.

- ◆ How do you change the parametric equations to get a horizontal translation?
- ◆ How do you get a stretch in the vertical direction?
- ◆ What happens to the curve if you edit it to “stretch” or “translate” the parameter t ?

(Answers on page 146.)

Lissajous fun

Old sci-fi television shows often showed Lissajous figures in their introductions. To make a Lissajous figure in the old, analog universe, you would use two tone generators and hook them up to the horizontal and vertical axes of an oscilloscope. This is exactly the same as using parametric equations the way we did above. The only thing we have to do is change the frequencies of the tones—the sine waves. Let's look at an example.

- ◆ Edit the equation to be:

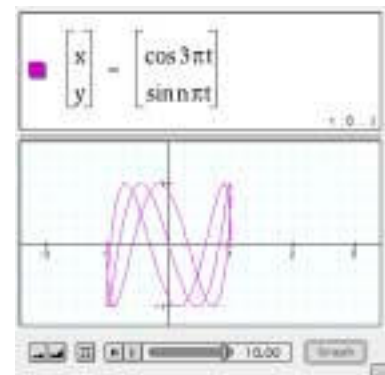
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(8\pi t) \\ \sin(10\pi t) \end{bmatrix}$$

We can play with that, changing the frequencies inside the vector to other values. But it's more fun with a slider.

- ◆ Edit the right-hand vector again:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(3\pi t) \\ \sin(n\pi t) \end{bmatrix}$$

- ◆ Change the **Slider Values** for **n** so that it goes from 0 to 10 in 40 steps.
- ◆ Press the animation button (the triangle) on the slider to watch the figure change.



Things to do

Investigate these questions and figure out why the curves work the way they do based on your understanding of the sine and cosine functions.

- ◆ Using the example from above (and the illustration), which values of n give “closed-loop” graphs, and which seem to have ends hanging out in the air? Which ones are symmetrical? What symmetry do they have?
- ◆ Set up a different animation, using these expressions:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(4\pi t + n) \\ \sin(2\pi t) \end{bmatrix}$$

- ❖ How is this different from what we were doing above? Why does it look the way it does?
- ❖ The expression above makes an animation that appears (to some people) to rotate around the x axis. What conditions do you think you need to make it rotate around the y axis? Make a conjecture, then use Graphing Calculator to perform experiments to test your conjecture.

Parametric equations without trig functions

Trigonometric functions are wonderful for parametric equations, but you can use any function you like for x or y .

- ❖ Set up Graphing Calculator to plot this parametric equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$$

- ❖ Choose **New Math Expression** from the **Math** menu.
- ❖ Try to write a “regular” function (i.e., $y = \text{something}$) that matches the curve you see in the window. Then explain why it is the way it is.

Hint: One way is to solve for t and substitute. To do this, you have to remember what that vector-notation stands for.

Making a spring: 3D parametric curves

When you made the **2-Vector**, you may have noticed the **3-Vector** in the **Math** menu. We can use 3-vectors to make three-dimensional parametric curves.

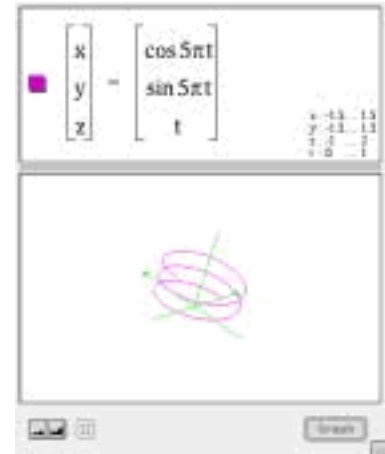
- ❖ Start with a fresh Graphing Calculator document.
- ❖ Choose **3-Vector** from the **Math** menu.

A new vector appears with x , y , and z .

- ◆ Type an equals sign and then make another new 3-vector, so you'll have something like this:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

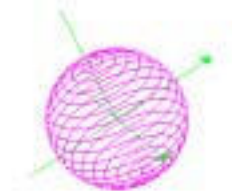
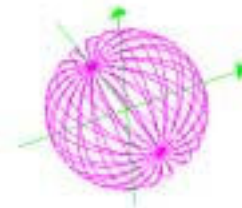
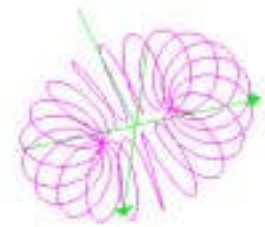
- ◆ Now edit the equation (just like before) to look like the picture, and graph the result:



The principle here is exactly the same as with the 2-dimensional parametric equations we've been doing. The difference is that as t increases, we add points in 3D-space to the curve, instead of just in the plane. So while the x - y image of the point is just going around in circles, it's steadily increasing in z . We get a helix, as you can see.

Things to do

- ◆ Explain why the spring in the picture goes around $2\frac{1}{2}$ times.
- ◆ What would the spring look like if you replaced the t with t^2 ? Predict first, then use Graphing Calculator to check.
- ◆ How would you make the spring longer?
- ◆ How would you make it have more turns?
- ◆ You don't have to use $\begin{bmatrix} x & y & z \end{bmatrix}$ for the left-hand vector. You can also use $\begin{bmatrix} r & \theta & \phi \end{bmatrix}$ (for spherical coordinates) or $\begin{bmatrix} r & \theta & z \end{bmatrix}$ (for cylindrical). See if you can make the pictures at right. The top one is in cylindrical coordinates; the bottom two are in spherical coordinates. (Solutions on page 147.)



Parametric Surfaces

Parametric curves need one parameter, t , because they are really one-dimensional. In fact, if you click on a 2D parametric curve to trace it, you see x and y of course, but you also see the value of t .

You need two parameters to make a parametric *surface* in three dimensions. Just as you use t as a parameter for a curve, use u and v as the parameters for surfaces. (This also explains why you're not allowed to use t , u , and v as sliders; Graphing Calculator would think that they were parametric-equation parameters.)

Note: Graphing Calculator 1.3 doesn't support 2-vectors or 3-vectors, so the material in this chapter won't work in that version.

◆ Set up this equation using 3-vectors:

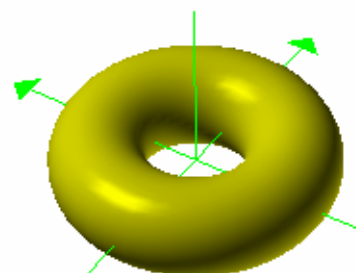
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

◆ Press **enter**.

You should see a “floating” square.

This is a simple surface, and we can do it with a regular function ($z = 1$). Let's set ourselves a task and walk through it step by step.

In two dimensions, we used trig functions to make a circle. Here, in three dimensions, let's make a *torus*—a doughnut-shaped 3D figure. Then we'll extend it. Just as we extended the 2D circle to be a spring (a helix), we'll turn our torus into a helix—and then a double helix.



Making the torus

We'll do this in several steps, altering our existing equations. We'll begin with a flat ring. In two dimensions, we had

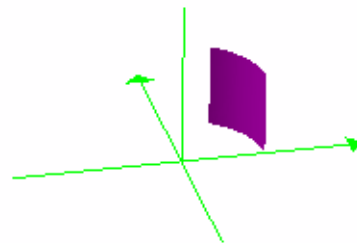
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

So we'll start by modifying that basic idea—cosines are horizontal, sines are vertical:

◆ Alter the equation so you have this:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos u \\ \sin u \\ v \end{bmatrix}$$

Again, there's an incomplete circle (a slice of a cylinder) because (as shown in the math pane) both u and v have ranges from 0 to 1.



Before we fix that, let's think about what's happening. For every pair of values for u and v , both in the range $[0, 1]$, Graphing Calculator computes three numbers: the x , y , and z coordinates. What v does is easy: it's the same as z . So all the points in a horizontal swath have the same z and therefore the same value of v . But u really acts as an angle (in radians of course)—the angle from the positive x -axis. On the surface, lines of equal u are vertical stripes, parallel to the z -axis.

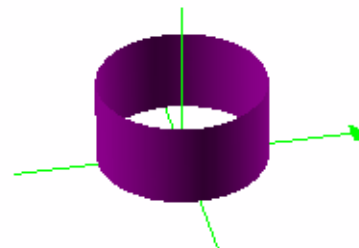
So you can think of u and v as coordinates *on the surface we're building*. That u is an angle is just a result of our using trig functions to get the curved surface we want. It could be anything—the functions we use determine how it gets mapped into 3D space.

We have to close the ring. We could do it by changing the range, but instead we'll fix the function.

◆ Change the equation to look like this:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos 2\pi u \\ \sin 2\pi u \\ v \end{bmatrix}$$

Now it's clearer that u goes around the ring, while v goes up and down. But for a torus, we want v to go around as well—but perpendicular to the ring. That will mean more trig functions.



Here's some ill-formed reasoning: add the cosine of v to the x and y components. That will make the ring fatter. This is a nice idea—and cosine, the horizontal part, is the right function—but it doesn't work. I'll save you the trouble and show you:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos 2\pi u + \cos 2\pi v \\ \sin 2\pi u + \cos 2\pi v \\ v \end{bmatrix}$$

The ring has been distorted and the u terms keep it going round and round. But instead of going around in the radial direction, the v terms move the circle back and forth. We want to make this v term be the distance from the z -axis, and we want it to oscillate between, say, $\frac{1}{2}$ and $1\frac{1}{2}$. To get that, we need

$$1 + \frac{\cos(2\pi v)}{2}$$

Instead of adding this item, we should multiply it. That is, we want to wiggle the radius of our ring by this varying factor. Edit your equation to look like the following and see if you get what we did:

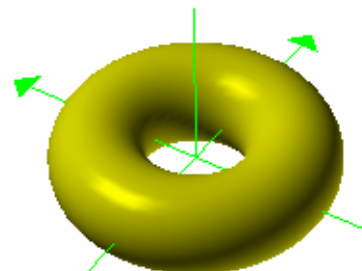
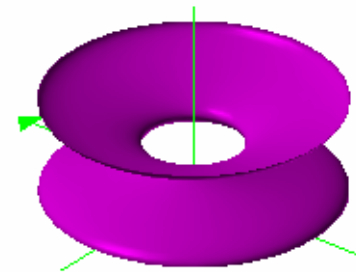
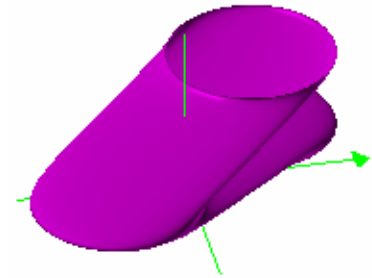
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos 2\pi u \left(1 + \frac{\cos(2\pi v)}{2} \right) \\ \sin 2\pi u \left(1 + \frac{\cos(2\pi v)}{2} \right) \\ v \end{bmatrix}$$

This still may look wrong, but we're on the right track. We just haven't done anything with z . Just as we used the cosine to give the ring some width horizontally, we need to use the sine of v to make the vertical part look right. and we'll divide by two to give it a circular cross-section:

◆ Once again, edit and graph:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos 2\pi u \left(1 + \frac{\cos(2\pi v)}{2} \right) \\ \sin 2\pi u \left(1 + \frac{\cos(2\pi v)}{2} \right) \\ \frac{\sin(2\pi v)}{2} \end{bmatrix}$$

And it looks good enough to eat.



Adding lines for understanding

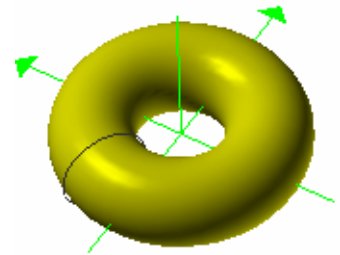
If you're tired—or if you completely understand how u and v can work together with vectors to make a donut—you can skip this section. But if you have some energy, stick around. We can use Graphing Calculator itself to help us figure out how this worked. In particular, we said up above that u and v were like *coordinates* on the surface. How does that work on a torus?

We're going to make some lines.

- ❖ Select the whole equation, both sides, and choose **Copy** from the **Edit** menu. (This will save us retyping it.)
- ❖ Choose **New Math Expression** from the **Math** menu. A new spot opens up.
- ❖ Choose **Paste** from the **Edit** menu. Now you have two copies of this huge block of equations. You may need to expand your window or scroll to work on the second one.
- ❖ Diversion: add 1 to the z component and graph it. Two donuts! Once you've seen it, get rid of the extra "1," but keep the equations.

Now we're going to use the new parametric equation to make a curve (not a surface) to show us what u and v do. So we'll need to use t .

- ❖ Edit the second equation so that all of the v 's become t 's and all the u 's become n 's.
- ❖ Give the curve a contrasting color.
- ❖ Change the range of n to be from 0 to 1 with 50 steps.
- ❖ Animate! If you have trouble seeing the circles, expand them slightly by editing the 2's in the denominators for the curve (all three of them) to be 1.9.



You should get something like the illustration. As this animates, the circle goes around, showing what happens as u changes (since u became n when we animated).

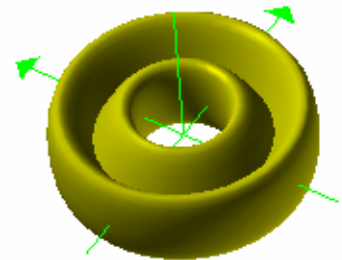
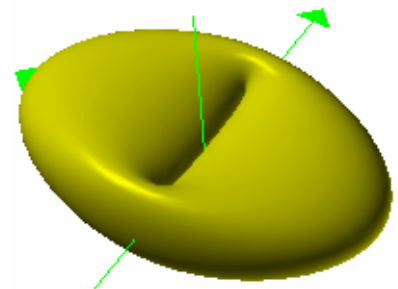
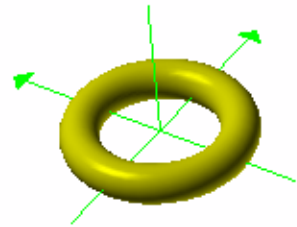
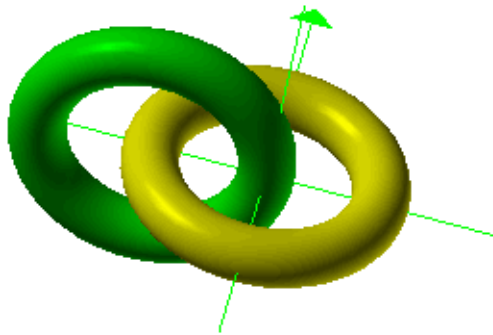
Things to do

Before we go on to making the [double] helix, here are some things you might try:

- ◆ Add another equation. Make it another parametric curve, this time showing values of v instead of u .
- ◆ Add another equation. This time make it a donut where the x axis goes through the donut instead of the z axis.
- ◆ The torus we made has cylindrical symmetry. It turns out that the equations are simpler in cylindrical coordinates. That is, find the right-hand vector for this setup, that makes the same torus that we made above:

$$\begin{bmatrix} r \\ \theta \\ z \end{bmatrix} = \begin{bmatrix} 1(u, v) \\ 2(u, v) \\ 3(u, v) \end{bmatrix}$$

- ◆ Some “modified” donuts appear at right. See if you can make them (Solutions on page 148)
- ◆ Make two donuts that link, like a chain:



Changing the donut into a double helix

We can change the donut into a helix in pretty much the same way that we turned the parametric circle into a helix. (See “Making a spring: 3D parametric curves” on page 82.)

Basically, we’ll use the parameter that goes around the torus the long way— u —to control the height, which is entirely in z .

- ◆ Start with a “clean” torus.
- ◆ Modify the z component by adding u . The equation and picture should look like the ones below (if you’ve switched to cylindrical coordinates, that’s fine):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos 2\pi u \left(1 + \frac{\cos(2\pi v)}{2} \right) \\ \sin 2\pi u \left(1 + \frac{\cos(2\pi v)}{2} \right) \\ \frac{\sin(2\pi v)}{2} + u \end{bmatrix}$$

- ◆ Make the helix go around one more time by changing the range of u to $-1 \dots 1$. Graph to see what that did.
- ◆ Make the tube skinnier by changing the three denominator 2 s to 4 s. Graph again.
- ◆ Make the helix steeper by changing the “ $+u$ ” we added to the z component to a “ $+3u$.” And graph again. Nice helix!

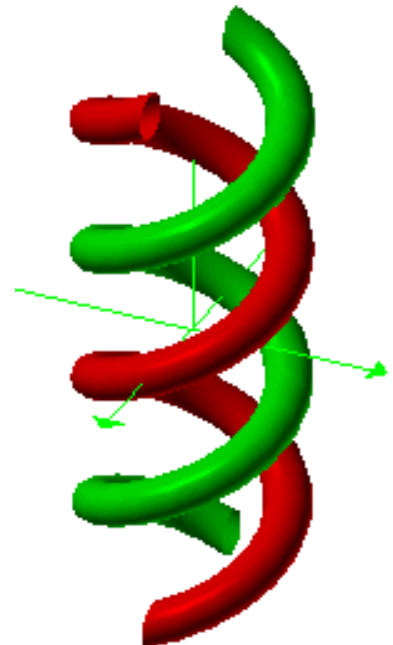
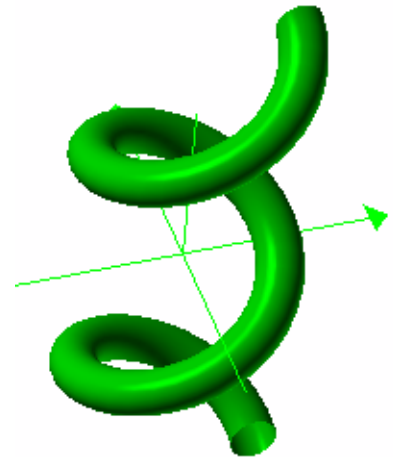
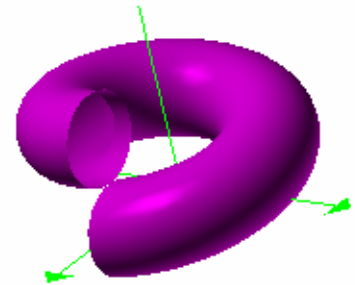
Now we want to make it a double helix. Our strategy will be simply to add another parametric surface. To save typing, we’ll start with the one we have and modify it.

- ◆ **Copy** the equation for the surface you now have.
- ◆ Choose **New Math Expression** from the **Math** menu. A new place appears, suitable for a new equation.
- ◆ **Paste** the equation in. When you graph it, the two helices will be right on top of each other. If the colors contrast, you’ll see “bleed-through.”

Now we need to rotate one of them. We want to change the x - y position of every spot to be a different place on the circle. That’s controlled by u .

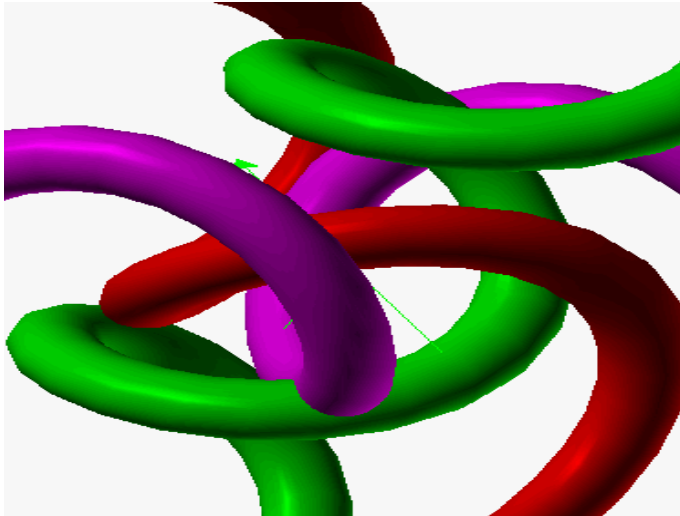
- ◆ In the new helix, modify the x and y components so that you add 0.5 to u in both expressions (but not in the one for z). Your equation and graph should resemble these:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos 2\pi(u + 0.5) \left(1 + \frac{\cos(2\pi v)}{4} \right) \\ \sin 2\pi(u + 0.5) \left(1 + \frac{\cos(2\pi v)}{4} \right) \\ \frac{\sin(2\pi v)}{4} + 3u \end{bmatrix}$$



Things to do

- ◆ Instead of putting that 0.5 in by hand, put in a slider, n , instead. Give it a range from 0 to 1 and see how it looks when it animates.
- ◆ Change the second helix so it's smaller, or larger, or goes in the opposite direction.
- ◆ Control more aspects of the helices with parameters. For example, use a named variable parameter such as b for that "4" in the denominator. Do the same for the helices' pitch (the three).



Fancy Stuff

Graphing Calculator has a number of features that can help you express your mathematics more flexibly or precisely. We've seen how you can use parametric equations—using vectors—as one example of this kind of thing. What else does Graphing Calculator have to offer? We'll discuss how you can define the color of your surfaces and how to wrap pictures around them. On the math side, we'll be learning about logical conditions, if statements (which make it easy to define piecewise functions), and summation notation. We'll also define functions and variables, do some calculus, and solve ordinary differential equations graphically. Finally, we'll dip our toes into vector fields.

Coloring

This is a really fun part of Graphing Calculator. You don't have to settle for the colors in the menu. You don't have to settle for one color for each surface. You can use mathematical expressions to color surfaces (and curves). And you do it with vectors.

Note: Graphing Calculator 1.3 doesn't support 2-vectors or 3-vectors, so the material in this chapter won't work in that version.

Here's the idea. After the formula that gives you the surface—regular, implicit, parametric, it doesn't matter—you put a comma and then a vector expression. This new expression tells Graphing Calculator the red, green, and blue values for the surface; you use the labels r , g , and b . These should take on values from 0 (black) to 1 (fully colored). anything less than zero is black; more than 1 is fully colored.

Let's try it:

- ◆ Start a fresh document.
- ◆ Enter this equation (with its comma-condition) and graph it:

$$z = 2 - x^2 - y^2, \begin{bmatrix} r \\ g \\ b \end{bmatrix} = \begin{bmatrix} 1 - x \\ 1 + x \\ z \end{bmatrix}$$



Note: If you're not using high resolution (see page 62), the colors may not look right.

Note: If you haven't read the last section on parametric surfaces, you may not know about the things in brackets. Choose **3-Vector** from the **Math** menu to get one. Edit it normally.

In the picture, x points right, y is into the page, and z is up.

If you're reading this on paper, you probably can't see the colors. The green part is to the right. In the equation, the green component is $1 + x$. So as x increases, g gets bigger. It fades in throughout the range (that is, g from 0 to 1) and so is fully green for all positive x .

So why is it bluish on the top? Because blue is proportional to z , and blue plus green equals cyan, that light blue color. Similarly, we see a yellow stripe (white on top) because the red component is increasing for negative x —and red plus green equals yellow (contrary to what we learned with paints).

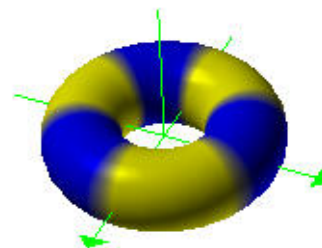
Let's see some more ideas.

Enter these and see what they look like. Notice how you can use different coordinate systems in the color vector.

$$z = 2 - x^2 - y^2, \begin{bmatrix} r \\ g \\ b \end{bmatrix} = \begin{bmatrix} \sin 2\pi x \\ \sin 2\pi y \\ \sin 2\pi z \end{bmatrix}$$

$$\sin 2\pi x + \sin 2\pi y + \sin 2\pi z = 2, \begin{bmatrix} r \\ g \\ b \end{bmatrix} = \begin{bmatrix} |\cos \theta| \\ \frac{z+2}{4} \\ |\sin \theta| \end{bmatrix}$$

- ◆ Use the torus we made above and a color vector (you can also use parameters in the color expressions) to make the picture at right.
- ◆ Make your double helix into a pair of striped snakes.



We've been keeping the functions for the colors pretty much independent and rather stark. But you don't have to; if the components have similar terms and a reduced range, you can make subtler color combinations.

◆ Try this:

$$z = x^2 + y^2 - 3, \begin{bmatrix} r \\ g \\ b \end{bmatrix} = \begin{bmatrix} \frac{z + 10}{10} \\ \frac{z + 8}{10} \\ \frac{2z + 2}{10} \end{bmatrix}$$

So far we've concentrated on using mathematics to describe interesting color treatments. Even better is to do the same in order to show more information on a graph than we would otherwise see. On the surface below, we use color to show the "snow line," the place where $z = 1$.



Conditions

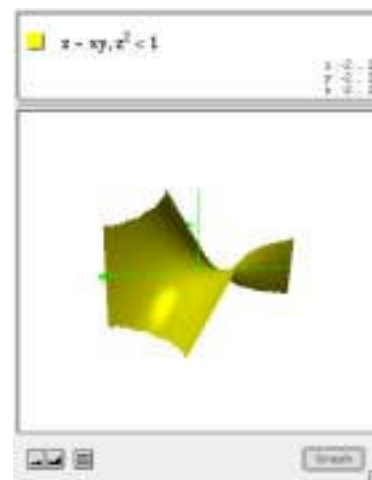
The way we used color vectors, with the comma, is really a special case of a more general Graphing Calculator feature: using *conditions*. If you follow an equation with a comma and a Boolean (true/false) expression, you get the plot for the region where the expression is true. You can see a simple example at right.

This works only in 3D graphs—Graphing Calculator will explain that it is “inappropriate” in a 2D environment.

If statements

Sometimes it's too hard to express what you're after in the domain of a single function. In that case, you may need an if statement. Let's see how this works.

- ◆ Start with a fresh document.
- ◆ Type $y = x^2$ if $x < 1$. Then graph the result.



You should see the window at right. Notice how the parentheses appear automatically, and how the function doesn't exist to the right of $x = 1$.

This works fine if you have a single piece of function, but what if you want a different function to the right?

◆ Put the cursor anywhere in the **if** construction.

◆ Insert a new clause in the **if** statement.

If you're using a Macintosh, press **control-A**. If you're using Windows, press **ctrl-shift-=**.

◆ In that new clause, enter $2 - (x - 2)^2$, and graph the result (shown).

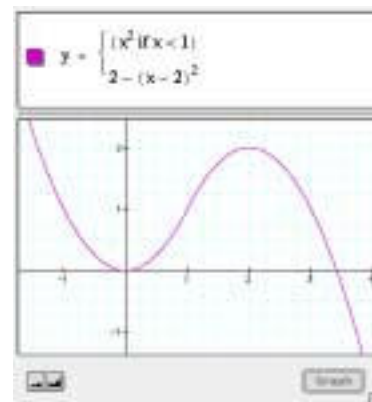
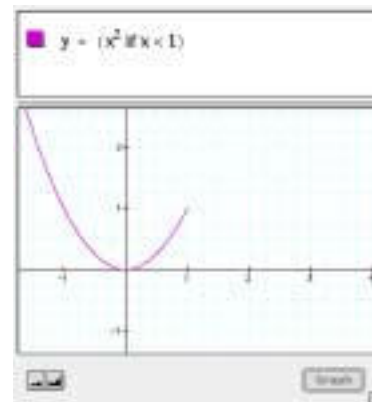
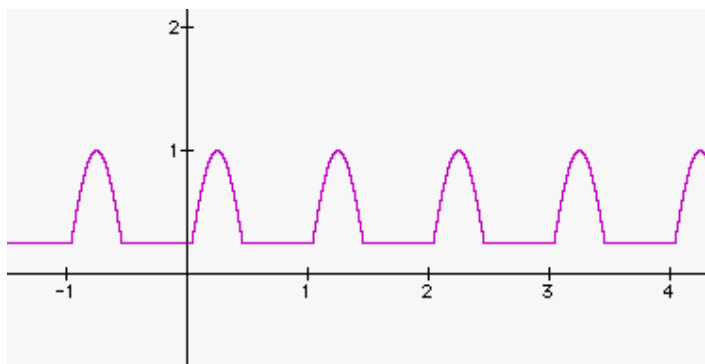
You should see the equation at right, brackets and all, which means, "y equals x squared if x is less than one, otherwise, y equals $2 - (x - 2)^2$."

You can make quite complicated graphs this way. Remember three things:

- You can add as many conditions as you like.
- Added clauses appear *after* the one the cursor is on.
- Graphing Calculator evaluates the conditions in order—if one of the early conditions is true, it won't even look at later ones.

Things to do

◆ See if you can make this graph:



- ◆ You can use this split-the-domain trick in 3D graphs as well. Try entering this one:

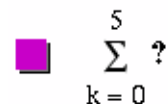
$$z = \begin{cases} 2 - y^2 & \text{if } (|y| < 1) \\ y & \text{if} \\ \frac{x^2}{4} & \end{cases}$$

Summations

Using the “big sigma” in Graphing Calculator is a lot of fun. It is easier in commercial versions than it is in version 1.3. Summations appear in the **Math** menu along with several other new features (such as integrals, which we’ll talk about soon). Let’s see how it works:

- ◆ Start with a fresh document.
- ◆ Choose **Summation** from the **Math** menu.

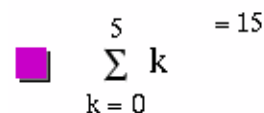
On the Macintosh, you can use the shortcut **control-S** (not **⌘-S**, which is **Save**). There is no shortcut on Windows.



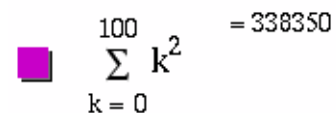
$$\sum_{k=0}^5 ?$$

You’ll see a default summation, as shown at right.

- ◆ Press **tab** repeatedly. Notice how the selection cycles though all of the places you might want to edit, including the question-mark. Stop there, with the question-mark selected.
- ◆ Type **k**. The value of the expression appears to the right as with any arithmetic computation.
- ◆ Tab some more until you get to the upper limit, the 5.
- ◆ Edit that to be 100. Be Gauss! What is the sum of the first hundred integers?
- ◆ Tab to the *k*. Type **^2** to square that *k*. You should see something like the picture at right.



$$\sum_{k=0}^5 k = 15$$



$$\sum_{k=0}^{100} k^2 = 338350$$

That’s fine for adding up series. But you can also add up series of functions.

- ◆ Enter this:

$$y = \sum_{k=0}^5 (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Zoom out so you can see what happens. Looks wiggly, right? Kind of like a sine curve except that it blows up after a few cycles.

- ❖ Change the upper limit to different values. See how you get more wiggles the higher the upper limit—that's because the sum has more terms, and they're alternating in sign because of the $(-1)^k$.
- ❖ Choose **New Math Expression** from the **Math** menu and enter $y = \sin x$ as your second function. Press **enter** to see them both together.
- ❖ Change the upper limit to n. Make sure it ranges from 0 to 10 in 10 steps, and animate it. Way cool. This is the Taylor series expansion for $y = \sin x$.

Things to do

Let's explore geometric series.

- ❖ Enter the following:

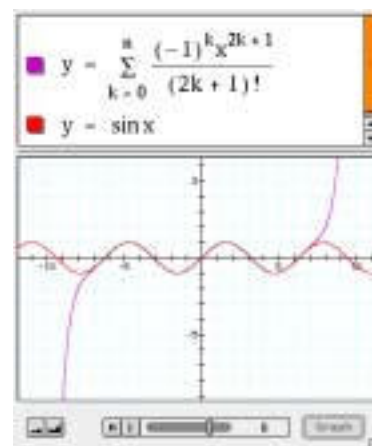
$$\sum_{k=1}^5 \left(\frac{1}{2}\right)^k$$

You should get 0.96875 as a result, which is correct. It's the same as $31/32$. (That is, $1/2 + 1/4 + 1/8 + 1/16 + 1/32$.)

- ❖ Change the upper limit from 5 to 100. You should now get a result of 1, which is not exactly correct, but is very very close. Graphing Calculator has rounded.
- ❖ What's the biggest number you can substitute for the "100"—the upper limit—and still have Graphing Calculator not round?
- ❖ Can you explain why a larger number of terms rounds up to 1?

Hint: look in the **Edit** menu under **Preferences**.

At any rate, the above, while not a proof, is pretty convincing evidence that the infinite sum



$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = 1$$

But what happens when the 1/2 is a different number?

- ❖ Try other numbers in place of the 1/2 in the expression, using 100 as the upper limit. Try 0.1, 0.2, 0.25, 0.4, 0.6, 2/3, 0.9, 1, 2, 10, and 10,000. Try negative numbers as well.
- ❖ Why do you suppose that numbers like 0.2 give clean, rounded results, while numbers like 0.9 do not? (Assume ten significant figures.)
- ❖ Can you guess a rule for the sum of the infinite series when the number is a fraction and the numerator is one less than the denominator? That is, what's

$$\sum_{k=1}^{\infty} \left(\frac{p-1}{2p}\right)^k?$$

- ❖ We can even get Graphing Calculator to show us the result for all different values of the thing inside the parentheses. Enter this function:

$$y = \sum_{k=1}^{40} x^k$$

That is, we're looking at the sum as a *function* of the number you take powers of. If you trace that function, you can see that, for example, when $x = 0.5$, $y = 1$. Now the actual exact result of the sum of an infinite geometric series is

$$\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}, \quad (-1 < x < 1)$$

- ❖ Compare your sum to 40 terms with theory by plotting the function $y = x/(1-x)$ along with the one you already have. What do you notice? Can you explain the differences?

Functions

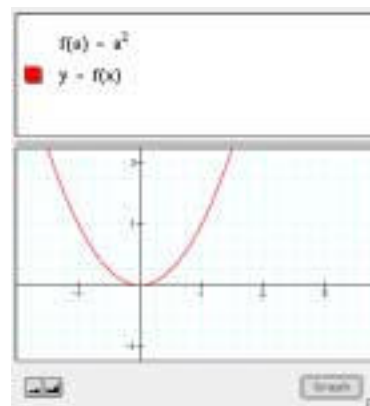
You can create functions in Graphing Calculator and then refer to them by name. But you have to be careful how you do it. Follow these instructions carefully! First we'll make a function that squares its argument.

- ◆ Start with a fresh document.
- ◆ Type **f**. This is the name of our function.
- ◆ Now type **control-9** (not **Shift-9**). A pair of parentheses appear.
- ◆ Type **a**. That's our function argument.
- ◆ Get to the right of the parentheses by clicking, using the right arrow, or simply pressing a close parenthesis (**shift-0**).
- ◆ Type **= a^2**. There! You've defined the function f to square its argument. Let's see if it works.
- ◆ Choose **New Math Expression** from the **Math** menu.
- ◆ Type **y = f**
- ◆ Type **control-9** to get the parentheses
- ◆ Type **x** to insert the argument.
- ◆ Graph the result. Your window should look like the one at right.

■ $f(|)$

■ $f(a|)$

■ $f(a) = a^2$



The key thing to remember—and it isn't easy—is that we use parentheses for two purposes:

- Grouping (and implied multiplication)
- Specifying function arguments

Graphing Calculator can't tell the difference from context alone. If you just write $f(a)$ with normal parentheses, you get default values of 1 for the two constants. So $f(a)$ is 1 times 1, or 1. Not what you want!

So when you define a function or use a function you have defined, you have to tell Graphing Calculator to use the special parentheses—which look the same as the ordinary ones. And you get the special ones by typing **control-9** instead of **shift-9**.

Note: You do not have to use the special parentheses when you use built-in functions such as `sgn()` or `log()`. It would be more consistent, but the designers feel that if you're far enough along to be using this advanced feature, you can handle it until they figure out something better.

Things to do


You can do great things with functions. Here are two ideas:

- ◆ Define a “combinations” function using the **ctrl-9** parentheses, as follows:

$$C(n, k) = \frac{n!}{(n - k)!k!}$$

(Note that you can have more than one argument.)

You can use this to do combinatorial calculations. For example, if you want to know how many handshakes there would be if everyone in a group of 10 shook hands with everyone else, you could define the function and then use it, as you can see at right:



$$C(n, k) = \frac{n!}{(n - k)!k!}$$

$$C(10, 2) = 45$$

You can explore composition of functions. For example, define and plot the following—and explain the result:

$$f(x) = 1/x$$

$$g(x) = \sin x$$

$$y = f(g(x))$$

$$y = g(f(x))$$

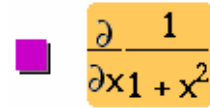
Symbolic calculus

If you're far enough along in your mathematical life to know about calculus, Graphing Calculator can help you out—in performing numerical calculations, symbolic calculations, and in learning what it's all about. You do have to be careful, though; these step-by-step instructions will introduce the features and give you some good how-to advice.

We'll start with derivatives. We'll study $y = \frac{1}{1 + x^2}$.

- ◆ Start with a new document.

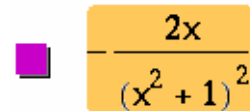
- ◆ Enter $\frac{1}{1+x^2}$. Careful—don't enter $y = \frac{1}{1+x^2}$.
- ◆ Put the cursor before the expression (or select the whole expression) and choose **Derivative** from the **Math** menu. You'll see this:



You may recognize this as a partial derivative. Graphing Calculator does total derivatives as well; they're in the same menu.

- ◆ But what *is* that derivative? Select the whole thing (if it isn't selected already) and choose **Simplify** from the **Math** menu.

The shortcut for **Simplify** is **⌘-y** on the Macintosh and **ctrl-y** on Windows.



You'll see the result:

Now, suppose you wanted to see the graphs of both functions. When Graphing Calculator figured out the derivative, it replaced the original function. So if you want them both, you need to do a little planning. So let's do it again...

- ◆ Start with a new document.
- ◆ Enter $\frac{1}{1+x^2}$.
- ◆ Copy that expression. (Use **Copy** on the **Edit** menu).
- ◆ Create a new math expression.
- ◆ **Paste** the original expression in.

The second one should still be selected.

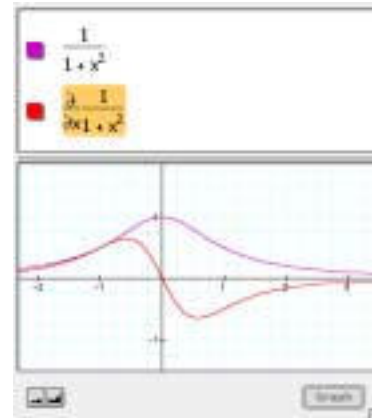
❖ Choose **Derivative** from the **Math** menu.

❖ Graph the result.

You'll see the function and its derivative.

Note two things:

- Graphing Calculator assumes that, if you just enter an expression in x , you really mean $y = \text{that thing}$ —and it will make the graph.
- You don't have to simplify the expression to get Graphing Calculator to graph it.



You can use the technique we outlined above—copying expressions and pasting them—to keep a record of the steps of your work.

The difference between total and partial derivatives

In beginning calculus, you tend to do total derivatives. But if you think about it, partial derivatives are easier to do, so they're the default. A little experiment will show you what we mean, and illustrate another technique:

❖ Create a new document.

❖ Enter $y = ax$.

❖ Select both sides.

❖ Choose **Derivative** from the **Math** menu.

You'll see something like the picture at right. Graphing Calculator uses an unorthodox but clear notation for “take the derivative of both sides.” You can use this with squaring, square roots, subtracting 2, and so on.

$$\frac{\partial}{\partial x} (y = ax)$$

❖ Choose **Expand** from the **Math** menu.

Important: Don't use **Simplify**! Try it if you want, but be ready to **Undo**. See if you can figure out why Graphing Calculator does this!

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x} ax$$

❖ Select the right-hand side and this time *do* choose **Simplify**.

So that's partial. Let's see what happens with a total derivative.

$$\frac{\partial y}{\partial x} = a$$

❖ Choose **New Math Expression** from the **Math** menu.

- ◆ Once again, enter $y = ax$ and select both sides.
- ◆ Choose **Total Derivative** from the **Math** menu.
- ◆ Once again, **Expand** what you have and **Simplify** the right-hand side. You'll see the result at right.

$$\frac{d}{dx}(y = ax)$$

$$\frac{dy}{dx} = a + x \frac{da}{dx}$$

Aha! Graphing Calculator realizes that a might be a function of x , so when you ask for the total derivative, it applies the chain rule. In beginning calculus, we ignore that possibility. We assume that a is a constant, so its derivative is zero and the second term vanishes.

Ordinary differential equations

If you have a commercial version of Graphing Calculator, you can plot solutions to first-order ordinary differential equations (ODEs) when you present them in the form

$$\frac{dy}{dx} = f(x, y)$$

Note: The exercises in this section won't work in Graphing Calculator 1.3.

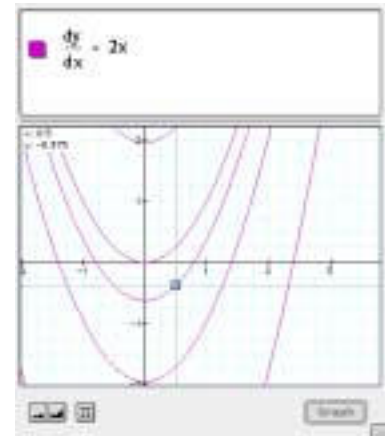
- ◆ Start with a new document.
- ◆ Enter the following:

$$\frac{dy}{dx} = 2x$$

Remember that we have to use a total derivative here; you'll find it in the **Math** menu. And under no circumstances use simple division to get the thing that looks like a fraction (the d 's will cancel)!

You should see a display like the one at right. Notice a few things:

- ◆ One of the curves looks kind of like $y = x^2$. That's good, because it's an obvious solution.
- ◆ Other solutions are showing, because this differential equation has many solutions that differ by a constant.
- ◆ If you click in the graph and drag (as shown) you can see solutions in between the ones Graphing Calculator shows you by default.

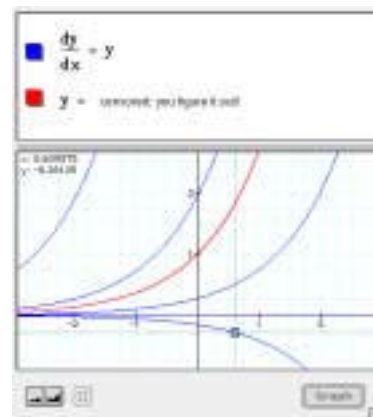


Of course, that was a pretty easy ODE, which is a good thing because they can get pretty hairy pretty fast.

Here's another one you may be able to solve:

$$\frac{dy}{dx} = y$$

Display it and see if you can plot an ordinary function with it that matches the curve, as shown at right:



On the other hand, a differential equation such as

$$\frac{dy}{dx} = x - y$$

is not so simple! Try entering it; probably, no functions leap to mind that solve that ODE.

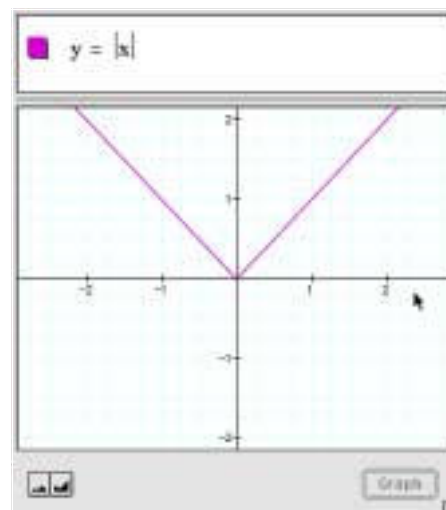
Integrals

Graphing Calculator does support integration, but it's a little perilous because the integration itself is done on (of all things) an integral server. So you have to be connected to the Internet, and the server has to be up. And the server has to understand the particular integral you're doing. So things may not work, but it's fun to try.

- ◆ Enter $y = x^2$ the usual way.
- ◆ Select the x -squared side.
- ◆ Choose **Integral** from the **Math** menu.
- ◆ Select the right-hand side.
- ◆ Choose **Simplify** from the Math menu.
- ◆ See (and graph if you like) the evaluated integral.

$$y = \int x^2 dx$$

$$y = \frac{x^3}{3}$$



Signum: a Useful Function to Know About

Graphing Calculator has a built-in function that's useful for creating graphs that look the way you want them to. You can discover it with calculus.

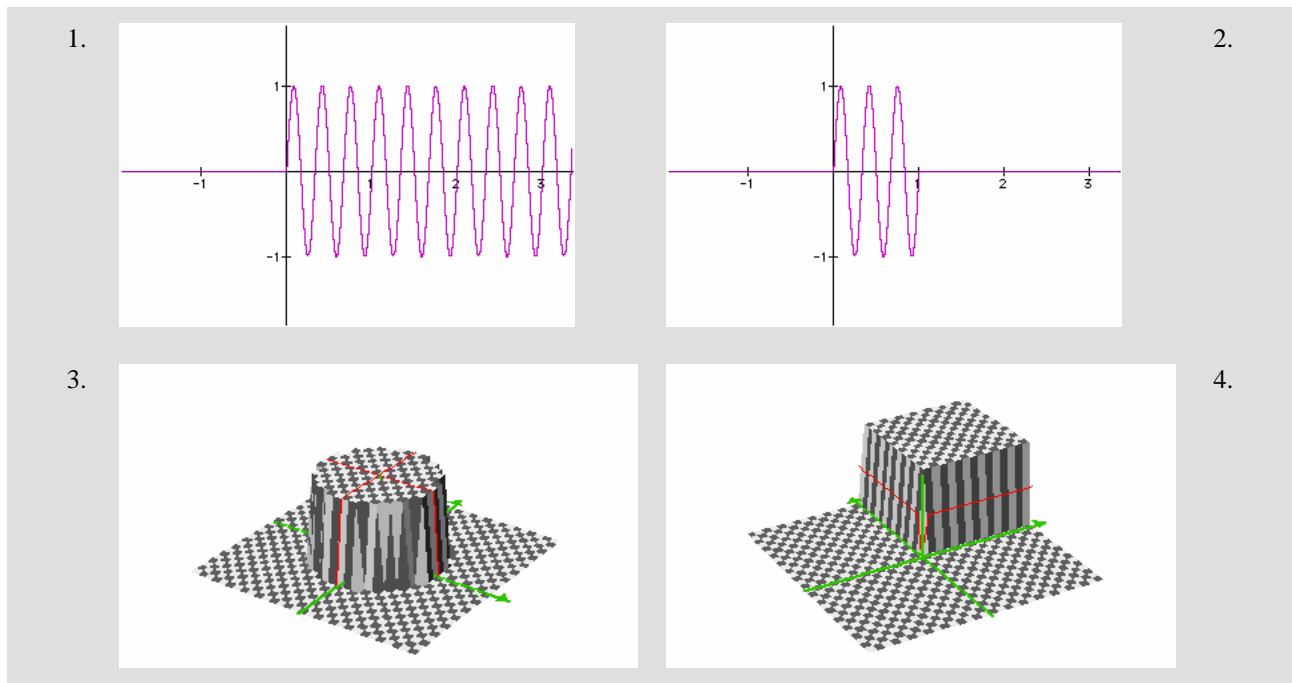
Think about this: What is $\frac{\partial}{\partial x} |x|$?

Predict first, and then find out using Graphing Calculator. Use the vertical bar key to make the absolute value marks.

You'll find the signum function, $\text{sgn}(x)$. It's especially useful in the form $\text{sgn}(x) + 1$ because then it's zero over much of its domain. You can use it to multiply by other functions to blank out the parts you don't want.

Things to do

- ◆ Write a good mathematical description of the signum function.
- ◆ Look up signum in a reference book. Is Graphing Calculator's signum function the same? (Look especially at $\text{sgn}(0)$).
- ◆ The first two graphs below are based on $y = \sin(6\pi x)$. See if you can figure out the equation that produced each graph.



Vector fields

Can we get a little insight—without taking a course in differential equations? It turns out that Graphing Calculator can graph vector fields—you can specify the components of a vector for every point in the plane and Graphing Calculator will display them.

Note: Graphing Calculator 1.3 doesn't support 2-vectors or 3-vectors, so the material in this chapter won't work in that version.

Since an ODE is a specification of the slope of a function at every point in the plane, we can make a *slope field*: a field of vectors having the right slope. The form of a vector field in Graphing Calculator is an equation like this

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

You substitute formulas for the question marks. Let's try it.

- ◆ Start with a new document.
- ◆ Enter the differential equation, $\frac{dy}{dx} = x - y$. Graph it.
- ◆ Make a **New Math Expression** (Math menu).
- ◆ Enter a **2-Vector**, an equals sign, and another **2-vector**.

Now we need to make the right-hand side of this vector function correct so that the vectors will have the right slope. Well, slope is rise over run, so let's make the run equal to 1, so the rise will just be the slope itself, $(x - y)$. That means that the vector equation will be

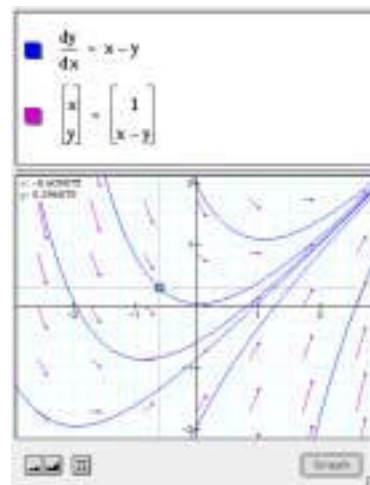
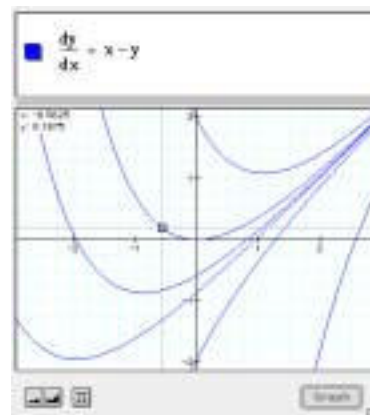
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ x - y \end{bmatrix}$$

- ◆ Change the vector equation to match the one we just figured out. As you drag in the window, you'll see something like the illustration at right:

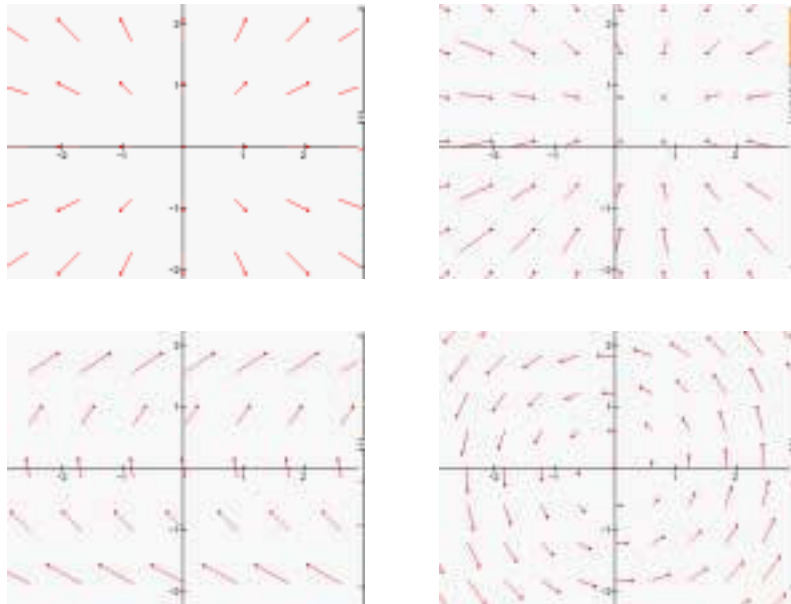
Notice how the functions go along with the vectors. They "flow" the way the vector point.

Things to do

Vector fields are an interesting way to look at functions; you can use them for a lot more than just visualizing differential equations.



See if you can make vector equations to match these fields.



(Solutions on page 149)

Note: You can use the resolution controls (page 62) to tell Graphing Calculator how fine to make the grid of vectors.

But wait! There's more!

If you're into physics or fluid dynamics, Graphing Calculator can help you with three-dimensional vector fields. It knows how to do dot and cross products—it even knows div, grad, curl (and all that).

Graphing Calculator has other displays of 3D functions that we haven't discussed: contour plots and density plots. And while we touched on color using the RGB system, you can also use the HSV (hue-saturation-value) system to map color onto your surfaces.

You can get information on these features from the Help system.

The people at Pacific Tech are adding more features to the program, so whatever I can do today may pale beside what you can do by the time you read this. But regardless of what's coming eventually, there's plenty to do with the software as it exists now. So have fun! Try stuff! Enjoy mathematics! And visit <http://www.pacifict.com/>—the Pacific Technologies website—often, to see what new things people have posted there.

Solving Problems with Graphing Calculator

In this section, we'll concentrate on Graphing Calculator techniques that, though they're about symbol manipulation, also have an impact on setting up problems and interpreting results. We'll also visit other disciplines and a few common genres of math problem.

Word Problems: the Big Picture

The typical algebra word problem starts with a situation: Amanda wants to figure out the height of a flagpole. It has some quantitative information: Its shadow is fifty feet long; she measures her own shadow to be six feet long, even though she's only five-six. In the course of the problem, you're supposed to express the situation as an equation, use algebra to change the equation to give you what you want, and then get the answer from the changed equation.

It's like trying to cross an impossibly busy street. You can't just cross the street, getting a direct answer; you have to go underground and take a tunnel—the rules of algebra—to get to the other side, and then you have to get back to the surface. Thus the solution has three parts:

Expressing the situation with symbols (going underground)

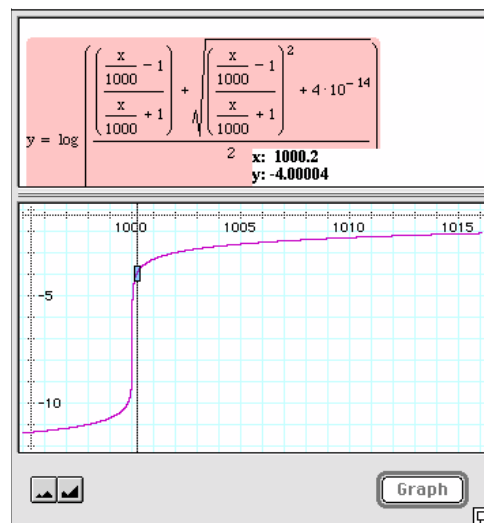
$$\frac{\text{Amanda's height}}{\text{Amanda's shadow}} = \frac{\text{Flagpole's height}}{\text{Flagpole's shadow}}$$

You have to understand the situation well enough to express it; here, you have to know the following:

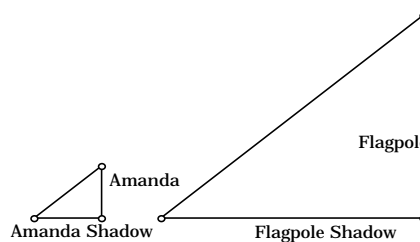
- That the length of a shadow is proportional to the height of the object.
- That it's the same proportion for all the objects
- How to set up a proportion. Often it helps to draw a picture.

Using symbol manipulation to change the equation (crossing under the street in the tunnels)

$$\text{Flagpole's height} = \frac{\text{Amanda's height}}{\text{Amanda's shadow}} \times \text{Flagpole's shadow.}$$



This is a graph of titration. It shows the log of the hydronium concentration as a function of the number of milliliters of 1 M HCl added to a liter of 1 M NaOH.



You need to know what you're looking for (the flagpole's height) and how to isolate it using the rules of algebra. Here, we multiplied both sides by the length of the flagpole's shadow and canceled, leaving the flagpole's height all by itself.

Interpreting the result (coming back to the surface)

The flagpole's height = $\frac{66 \text{ inches}}{72 \text{ inches}} \times 50 \text{ feet} = 45 \text{ feet}$.

In this case, "interpreting" means plugging in the numbers, calculating, and checking the answer for reasonableness. Amanda is shorter than her shadow; the flagpole is shorter than its shadow, too. That makes sense.

Graphing Calculator can help you with the symbol manipulation and with plugging in numbers—but it won't help a bit with figuring out what you want and setting up the equations. Too often we think that slinging symbols around is all there is to "doing algebra." It is a big part, sure, but it's only one out of three. Expressing the situation and interpreting the results are often more important. Use Graphing Calculator for symbol manipulation, so you can pay more attention to the main show—setting up problems, creating mathematical models, and seeing if your results make sense.

What Symbol Manipulation Is For

Somebody said that algebra is generalized arithmetic. It's how to do arithmetic with variables (such as x) as well as with constants (such as 3). So when you say $2x + 2x = 4x$, that's algebra; when you say $2 + 2 = 4$, that's just ordinary arithmetic. It's *generalized* because the x can stand for any number, so $2x + 2x = 4x$ could mean $2 + 2 = 4$ (if $x = 1$) or $12 + 12 = 24$ (if $x = 6$) or $-5 - 5 = -10$ (if $x = -2.5$), and so forth.

You use the rules of algebra to *transform* a symbolic statement into another one that you can use more easily. For example, suppose you know that the circumference of the earth is 40,000 km. And suppose you want to know its radius. What should you do?

Maybe you remember that the circumference of a circle (we'll call it C) is related to the radius (R) by a formula:

$$C = 2\pi R$$

That's fine if you know the radius and want the circumference, but our situation is reversed. We want to know R , so we want something that begins

$$R =$$

We get it through algebra, by transforming the thing we know ($C = 2\pi R$) into a different expression:

$$R = \frac{C}{2\pi}$$

This last expression gives the same relationship between C and R —*it's just changed into a form that fits our situation.*

That's all algebra is—changing equations and expressions into more useful forms. Often, an “algebra problem” involves *solving* an equation. That means getting something (usually x or y) on one side of the equals sign and everything else on the other. In the example, we solved the equation for R . Remember: Exercises in an algebra textbook that occur “naked”—without situations (e.g., Solve the following equations for x ...)—are only the middle part of the problem-solving process.

Dimensional Analysis

Here is a technique that may give you more ideas about how to use Graphing Calculator. Suppose we want to figure out the number of seconds in a year. This quantity shows up all the time in what we might call “back-of-the-envelope” questions, which themselves range from riveting to frivolous—or both. They include questions like these:

- ◆ How long would it take you to read the phone book aloud?
- ◆ How many times will your heart beat in your lifetime?
- ◆ How far is a light-year?

In the traditional method, as you might learn in a physics class, you write all the units—meters, seconds, ergs, whatever—that appear with the numbers in a calculation. You cross out like units in the numerator and denominator; they cancel algebraically as if they were variables.

The number-of-seconds-in-a-year calculation might look like this:

$$\begin{aligned} & 60 \frac{\text{seconds}}{\text{minute}} \times 60 \frac{\text{minutes}}{\text{hour}} \times 24 \frac{\text{hours}}{\text{day}} \times 365 \frac{\text{days}}{\text{year}} \\ & = 60 \text{ seconds} \times 60 \frac{\text{minutes}}{\text{minute}} \times 24 \frac{\text{hours}}{\text{hour}} \times 365 \frac{\text{days}}{\text{day}} \times \frac{1}{\text{year}} \\ & = 31,536,000 \frac{\text{seconds}}{\text{year}} \end{aligned}$$

Graphing Calculator will help you do this and keep it straight.

There are two ways, both basically the same. The first is more official; the second is just as good and faster, as long as you don't mess up. We'll do them both.

Method 1: Multiplying by 1

Dimensional analysis is all about converting units from one system to another. We have years. We want seconds. So we start with one year and try to re-express the quantity in seconds. We get there by repeatedly multiplying by one and canceling the leftover units, as in

$$1 \text{ year} = 1 \text{ year} \times \frac{365 \text{ days}}{1 \text{ year}}$$

The fraction is equal to 1, because 1 year *is* 365 days. Cancel the "year," and you get 1 year = 365 days. That's obvious. Then do the same to days, converting to hours (1 day = 24 hours), and you get

$$1 \text{ year} = 365 \text{ days} \times \frac{24 \text{ hours}}{1 \text{ day}}$$

The days cancel and you get 1 year = 8760 hours.

Here's a common mistake:

$$1 \text{ year} = 365 \text{ days} \times \frac{1 \text{ day}}{24 \text{ hours}}$$

This is correct, but not useful *because the units don't cancel*. You get

$$1 \text{ year} = 15.21 \frac{\text{day}^2}{\text{hour}}$$

and square days are not a common unit. The extra work of carrying the units around is worth it, because it acts as a check.

In Graphing Calculator, you can type in the units as well. But here's a catch. Let's just do the first step:

- ◆ Type **1 year * 365 days / year** .

And you'll get

$$1\text{year}365\text{day}\frac{\text{s}}{\text{year}}$$

You're already in trouble.

- ◆ Select the whole thing and choose **Simplify**, and you'll see:

365adsy

Which is pretty good, but not what we had in mind. Use single-letter abbreviations for units. Be sure they are unique, though Graphing Calculator is case-sensitive: If you use *y* for yards, you may use *Y* for years.

Here is the whole seconds-per-year thing in Graphing Calculator. Be sure to press the right arrow key to exit each fraction. It starts like this:

- ◆ Type **1 y = 1 y * 365 d / y (right arrow) * 24 h / d (right arrow)** etc.

You should see:

$$1y = 1y365\frac{\text{d}}{\text{y}}24\frac{\text{h}}{\text{d}}60\frac{\text{m}}{\text{h}}60\frac{\text{s}}{\text{m}}$$

Notice how the times sign goes away when it's not needed algebraically. It does look a little funny.

- ◆ Select the whole expression and choose **Simplify**, and you'll see:

$$y = 3.1536 \times 10^7 \text{ s}$$

Which is the right answer. Graphing Calculator cancels the units for you.

Method 2: Direct Substitution

This is faster and cooler, but treacherous.

- ◆ Type 1 y.

1y

- ◆ Select the y.

1y

- ◆ Type 365 d.

1· 365 d

Remember, *whenever something is selected, and you type, your typing replaces the selection*. Since you replaced y—the year—with its equivalent—365 days—the expression has the same value. Notice that Graphing Calculator doesn't just append the digits (making 1365, which is wrong)—it sticks in a little multiplication dot right where it belongs. Next step:

- ◆ Select the d.

1·365d

- ◆ Type 24h.

1· 365· 24 h

You can continue this way until you get to seconds:

1·365·24h

1·365·24·60m

1·365·24·60·60s

- ◆ Finally, select the whole thing and choose **Simplify**, and you'll see:

$3.1536 \times 10^7 \text{ s}$

Note: That opening number, 3.15, ought to look familiar. Close to π , right? You're not too far off if you remember that there are $\pi \times 10^7$ seconds in a year.

Using Substitution

You may remember that 60 miles per hour is 88 feet per second. You don't? Let's prove it.

- ◆ Start with 60 miles per hour and just for form, set it equal to itself. Use m and h , like this:

$$60 \frac{m}{h} = 60 \frac{m}{h}$$

Now substitute for the m —use 5280 feet. *Remember, when you substitute, substitute only what's equivalent.* A mile is 5280 feet.

- ◆ Select the m (on the right; leave the one on the left alone, so we remember where we started) and type **5280f**:

$$60 \frac{m}{h} = 60 \frac{m}{h} \text{ becomes } 60 \frac{m}{h} = 60 \frac{5280f}{h}$$

For hours, we select the h on the bottom on the right and substitute 60 minutes, and then 60 seconds, or maybe we just know there are 3600 seconds in an hour. Eventually, you get something like this:

$$60 \frac{m}{h} = 60 \frac{5280f}{60 \cdot 60s}$$

- ◆ Select the whole thing and simplify, and you get

$$\frac{60m}{h} = \frac{88f}{s}$$

Ta-daa!

Figuring Average Speed

Eloise, the fastest runner at Echo Valley High, runs the 100-meter dash in 12.1 seconds. Find her average speed in miles per hour.

Solution: The average speed is the distance divided by the time. So we put that information into Graphing Calculator. We'll use m for meters and plan on using M for miles.

❖ Write it twice, like this:

$$100 \frac{m}{12.1s} = 100 \frac{m}{12.1s}$$

❖ Now we'll convert the right-hand side. First let's do the seconds.

There are 3600 seconds in an hour, *but going this direction, we can't just substitute*. If we select the s , we have to type something equivalent to it (not, for example, $3600h$). Instead, we multiply by 3600, putting the seconds on the top so they will cancel the existing seconds on the bottom.

$$100 \frac{m}{12.1s} = 100 \frac{m}{12.1s} 3600 \frac{s}{h}$$

❖ Now convert meters to miles.

Use $1.6 \text{ km} = 1 \text{ mile}$ (which is good enough). First, convert to kilometers (use K). We can multiply by 1, or directly substitute $.001 K$ for our $1 m$.

$$100 \frac{m}{12.1s} = 100 \frac{0.001K}{12.1s} 3600 \frac{s}{h}$$

Now—if we cancel units—we've made it to kilometers per hour. Time to go to miles.

❖ Use capital M for miles, remember, and multiply by 1.

$$100 \frac{m}{12.1s} = 100 \frac{0.001K}{12.1s} 3600 \frac{s}{h} 1 \frac{M}{1.6K}$$

❖ Now **Select All** and **Simplify**.

$$\frac{100m}{12.1s} = \frac{360M}{19.36h}$$

Swell. We know she can go 360 miles in 19.36 hours if she keeps sprinting, but that's neither likely nor useful. We have to separate the numbers from the letters on the right. Here's a step-by-step:

- ◆ Select the M and grab it.

$$\frac{100\text{m}}{12.1\text{s}} = \frac{360\text{M}}{19.36\text{h}}$$

- ◆ Drag it farther to the right.

$$\frac{100\text{m}}{12.1\text{s}} = \frac{360}{19.36\text{h}} \text{M}$$

- ◆ Do the same for h .

$$\frac{100\text{m}}{12.1\text{s}} = \frac{360}{19.36} \text{M} \frac{1}{\text{h}}$$

- ◆ Now select the fraction and **Calculate**.

$$\frac{100\text{m}}{12.1\text{s}} = \frac{360}{19.36} \text{M} \frac{1}{\text{h}} \text{ becomes } \frac{100\text{m}}{12.1\text{s}} = 18.595 \text{M} \frac{1}{\text{h}}$$

When you adjust the units using drag algebra, you will see:

$$\frac{100\text{m}}{12.1\text{s}} = \frac{18.595\text{M}}{\text{h}}$$

So she was going over 18 miles per hour, on the average.

Things to do

Here are some exercises that use dimensional analysis and substitution.

- ◆ How fast is 55 mph in meters per second? If you could drive around the earth's equator at 55 mph, how long would it take? (Did you know that a *meter* was originally defined so that 10,000 km was the distance from the equator to the North Pole?)
- ◆ How many acres does a sheet of binder paper cover? (Use 8.5 X 11 inches; also, there are 640 acres in a square mile.)

- ◆ How fast is a furlong per fortnight? (There are 8 furlongs in a mile; a fortnight is 14 days.)
- ◆ Suppose you had a metric ton (1,000 kg) of water in a sphere. How big would the sphere be? Water is practically incompressible and has a density of 1 gram per cubic centimeter. (This question is relevant if you're making a zero-gravity swimming pool, and possibly in other situations.)

More About Symbol Manipulation

Graphing Calculator's built-in commands—**Expand**, **Simplify**, and **Isolate Term**—are great, but:

- You have to know when to use them.
- They can't do everything. You still have to know about algebra!

There are three rules of algebra:

- You can always multiply by 1.
- You can always add zero.
- If you do anything else, you have to do the same thing to both sides.

Adding zero is easy—just add whatever you want to add and its negative. And we've talked about multiplying by 1 when you convert from one kind of unit to another. You also multiply by 1 when you find common denominators. For example, suppose you want to add

$$\frac{1}{2x} + \frac{1}{2y}$$

In Graphing Calculator, you can't just select and **Simplify**. (It works if you have just numbers, but the variables mess things up.) Instead, we can multiply each term by 1 in order to construct a common denominator, in this case, $2xy$. Remember, if we select a term and then use an operator (such as $*$), the operator will apply to the selection.

- ◆ Type in the equation and select the left term.

$$\frac{1}{2x} + \frac{1}{2y}$$

- ◆ Then type $*y/y$. You'll see:

$$\frac{y}{y} \frac{1}{2x} + \frac{1}{2y}$$

If you **Simplify** now you'll be back where you started.

- ◆ Drag the bottom y into the fraction with the x . We're making the common denominator (See, you *do* have to know about algebra).

$$y \frac{1}{y2x} + \frac{1}{2y}$$

- ◆ Do the same to the right-hand term, except multiply by x/x .

$$y \frac{1}{y2x} + x \frac{1}{x2y}$$

- ◆ Now select one of those common denominators and drag it sideways until it distributes out.

$$(y + x) \frac{1}{y2x}$$

- ◆ Finally, **Select All** and **Simplify**.

$$\frac{y + x}{2xy}$$

Adding Fractions: the Shortcut

While the preceding technique is often useful, there's a faster way that applies to this particular situation.

- ◆ Start with the fraction. Grab one denominator and slide it until it distributes out.

$$\frac{1}{2x} + \frac{1}{2y} \quad \text{becomes} \quad \left(1 + \frac{x}{y}\right) \frac{1}{2x}$$

- ◆ Then grab the one that's left and slide it to meet the first.

$$\left(1 + \frac{x}{y}\right) \frac{1}{2x} \quad \text{becomes} \quad (y + x) \frac{1}{y \cdot 2x}$$

◆ **Select All** and **Simplify**, and you're done!

Operating on Whole Equations: the 100-Meter Dash Revisited

You can use the select-and-use-an-operator technique to do the same thing to both sides of an equation. An example will help. Remember this situation? We had to get the M and h out of the right-hand fraction to evaluate it. We grabbed and dragged; here's another solution.

◆ Select the whole expression.

$$\frac{100\text{m}}{12.1\text{ s}} = \frac{360\text{M}}{19.36\text{h}}$$

◆ Then type $*h/m$. You'll see:

$$\frac{h}{M} \left(\frac{100\text{m}}{12.1\text{ s}} = \frac{360\text{M}}{19.36\text{h}} \right)$$

We want to cancel the M/h on the right-hand side. This is unorthodox notation, but it makes sense. Imagine that the h/M term distributes across equals signs just as it does across addition. How can we use this?

◆ Select the whole thing and choose **Simplify**. You'll see:

$$\frac{100\text{hm}}{12.1\text{ Ms}} = \frac{360}{19.36}$$

Graphing Calculator cancels the units on the right and puts them where they belong on the left. Now, as before, we can just calculate the fraction and move the units back—either by dragging or by multiplying both sides by M/h .

One More Example: Einstein Was Here

In special relativity, the equation for a relativistic factor is

$$f = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

where v is your velocity (speed) and c is the speed of light. You can **Isolate** any term you want, but suppose you want to square both sides in order to get rid of the radical. Here's how:

$$\left(f = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \right)^2$$

- ◆ Select both sides.
- ◆ Type ^2.

You can see the result in the margin. What a hairy equation! The equals sign inside the parentheses is unorthodox, but you can resolve it when you

- ◆ **Select All** and choose **Simplify**.

We've removed the radical, as you can see in the lower, selected picture.

$$f^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$

Things to do

- ◆ Verify the following by turning the left-hand side into the right-hand side using Graphing Calculator:

$$1. \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} \qquad 2. \frac{1}{1 + \frac{1}{g}} = \frac{g}{1 + g}$$

$$3. \text{ Show that } \frac{1}{m} = \frac{1}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \text{ means that } m = \frac{3abc}{bc + ac + ab}.$$

- ◆ Bobby's slot car went 25 meters in 3.2 seconds. How many meters per second is that? How many miles per hour? What year was it?
- ◆ **You Can Be Einstein, Too.** Use the equation for the relativistic factor above. You will double your mass when $f = 2$. How fast (v) would you be going? The speed of light, c , is 300,000 kilometers per second, or 186,000 miles per second.

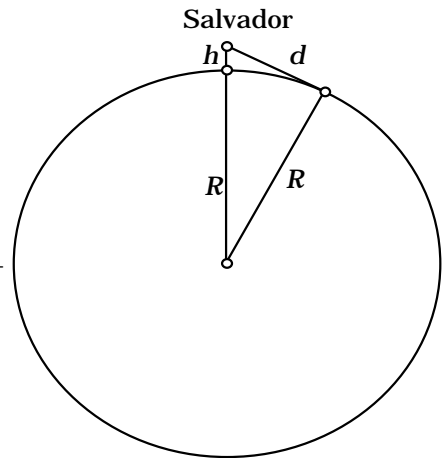
Two Extended Examples

Salvador and the Cliff

Here's a traditional problem:

Salvador stands on a cliff overlooking the sea on a clear day. As he stands, his eyes are 100 meters above the surface of the ocean. How far away is his horizon?

It's usually good to start by drawing a picture. The figure (produced with The Geometer's Sketchpad) shows R , the radius of the earth; h , Salvador's height above the sea; and d , the distance he can see—the length of the segment tangent to the earth's surface. Of course, the figure is not to scale, but the geometry is correct.



How do we get the answer using Graphing Calculator? Even if you can do it by hand, Graphing Calculator will help you avoid mistakes. And since you're sitting at a computer, you can record the steps you go through and print them out when you're done. We'll do it from the beginning. Skip as many steps as you like.

First, we recognize the right triangle in the diagram, so we start out with

$$a^2 + b^2 = c^2$$

But which is which? The two legs are R and d , and the hypotenuse, c , is $(R + h)$. To substitute, select the variable—without its exponent—and type what you want to substitute.

First we'll do the letters; we'll do the actual values later. Let's substitute $(R + h)$ for c step-by-step:

- ◆ Select c .

$$a^2 + b^2 = c^2$$

- ◆ Type R .

$$a^2 + b^2 = R^2$$

- ◆ Type $+$.

$$a^2 + b^2 = (R + |)^2$$

Notice that Graphing Calculator inserts parentheses, knowing that you're substituting for c . This feature is useful in lots of situations. And you didn't even have to close the parentheses.

- ◆ Type **h**.

$$a^2 + b^2 = (R + h)^2$$

- ◆ Now substitute for a and b .

$$R^2 + d^2 = (R + h)^2$$

You might have started right here. Now, what do we want to find? The distance to the horizon, d .

- ◆ Select d .

$$R^2 + d^2 = (R + h)^2$$

- ◆ Choose **Isolate Term** from the **Math** menu.

$$d = \sqrt{(R + h)^2 - R^2}$$

If you want, you can even select the whole thing (or just the right-hand side).

- ◆ Choose **Expand**.

$$d = \sqrt{2Rh + h^2}$$

If we're just interested in numbers, we can substitute. The radius of the earth is about 6400 km. The height, h , is 100 meters. You could do the conversion in your head, but we'll put the units in here for demonstration purposes.

- ◆ Select each variable and type in the value *and its units*.

$$d = \sqrt{2 \cdot 6400\text{k}100\text{m} + (100\text{m})^2}$$

- ◆ If you **Select All** and **Calculate**, you get:

$$d = \sqrt{1.28 \times 10^6 \text{ km} + 10000\text{m}^2}$$

Whew! Glad I put in the units; it reminded me I hadn't converted kilometers to meters yet.

Note: Don't mistake "km" for kilometers—that's *k* for kilometers and *m* for meters.

Now we have to substitute in for *k*. That's 1000 *m*.

- ◆ Do the substitution.

$$d = \sqrt{1.28 \times 10^6 \cdot 1000 \text{ m} + 10000 \text{ m}^2}$$

Again, not millimeters!

- ◆ **Select All and Calculate.**

$$d = \sqrt{1.28001 \times 10^9}$$

The meters come out. That's good—we want our answer in meters, don't we? But what about that root? Sometimes Graphing Calculator can't do it all at once.

- ◆ **Calculate** again.

$$d = 35777.2 \text{ m}$$

Salvador can see almost 36 km.

Unfortunately, if we want to put him up 200 m, we have to go back to the equation. Maybe we need a function and a graph.

Using a Graph to Help

Great idea.

- ◆ Go back to here:

$$d = \sqrt{2Rh + h^2}$$

Do the whole thing in kilometers so you don't have to convert.

- ◆ Substitute 6400 for *R*.

$$d = \sqrt{2 \cdot 6400h + h^2}$$

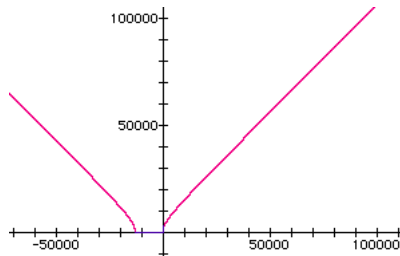
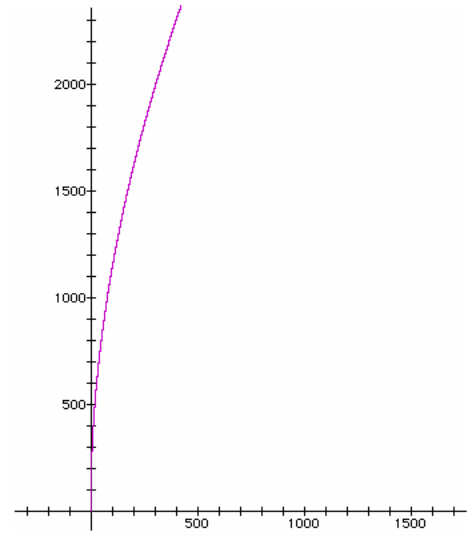
This time we'll leave out the units, so Graphing Calculator will actually graph it. Now we need x and y . We want to find d as a function of h , so that means d is y and h is x .

◆ Select and substitute!

$$y = \sqrt{2 \cdot 6400x + x^2}$$

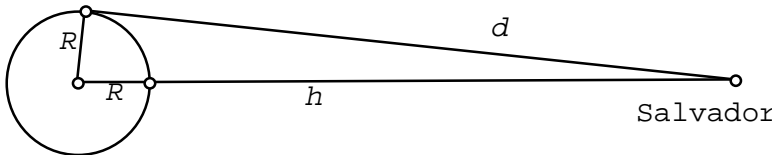
Be sure to substitute both x 's and don't lose the exponent! Now graph the equation. You may not see anything. The graph is off scale. You'll have to zoom out quite a bit, as you see in the margin.

Graphing Calculator can scale one axis differently from the other. (On the Macintosh, hold down **option** or **⌘** when you zoom; on Windows, hold down **shift** or **ctrl**.) But you can still see the shape of the curve, and if you zoom out *more*, you'll see:



For large values of x the curve looks a lot like $y = x$, the line that goes through the origin. What does that mean?

If you get really high above the earth, the distance to the horizon is about the same as your height. That makes sense, since the radius of the earth is only 6400 km, and the scale here is to 100,000 km; it's like holding an orange at arm's length and asking whether it's closer to the middle or the edge. It's closer to the middle, but not by much.



But what about the curve to the *left* of the y -axis? What does that mean? We'll let you figure it out yourself!

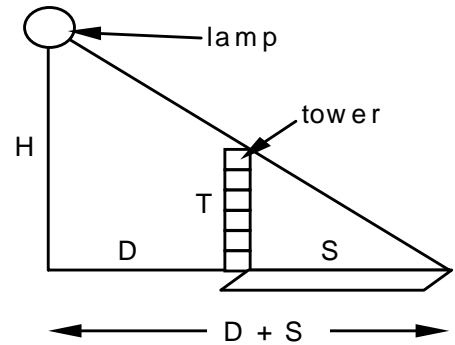
Amanda's Shadow Investigation

Our second extended example returns to Amanda and shadows. (See "Amanda's Shadow Investigation" on page 124.)

Amanda decides that the sun is too uncontrolled a light source, so she gets a lamp. She also decides that a flagpole is not only too tall, it's always the same height. So she gets a bunch of blocks. She plans to make towers out of blocks and use the lamp to cast the shadows. Before she actually uses the blocks, she decides to make a prediction.

She makes a diagram and writes a list of variables on paper or in a word processor:

- T The height of the tower
- H The height of the lamp
- D The distance from the lamp to the tower
- S The length of the shadow



She knows all about similar triangles, so she starts up Graphing Calculator and writes this equation:

$$\frac{T}{S} = \frac{H}{D + S}$$

Now she makes some measurements. The lamp is 150 cm tall. She has it set up 190 cm from the tower. She decides that first she'll vary the height of the tower and see how the length of the shadow depends on it. (She could have varied the distance to the lamp.)

She wants a graph to help her see what's going on. Graphing Calculator will graph a function only with special letters (such as x and y). She rewrites her list:

- T The height of the tower x
- H The height of the lamp 150 cm
- D The distance from the lamp to the tower 190 cm
- S The length of the shadow y

She chose x for the height of the tower (the *independent* variable) because that's what *she* will change. The length of the shadow, y , is

the *dependent* variable because it *depends* on the height of the tower. She could have done it the other way, but it's a tradition that y , the vertical axis, is the thing that depends on x , the horizontal, not the other way around. Next, in Graphing Calculator, she substitutes in by selecting and typing over the selection:

$$\frac{x}{y} = \frac{150}{190 + y}$$

Now for the algebra. She has to solve the equation for y . She's not sure exactly how to do it, so she selects the denominators and drags them to the other side with the hand, trying to get the y 's on top.

Dragging the denominators *up* doesn't work, just as it doesn't make sense algebraically. Dragging the denominator to the other side is the same as multiplying through.

$$x(190 + y) = 150y$$

Using **Expand** on the left-hand side gives

$$xy + 190x = 150y$$

Rearranging terms with the hand-grabber tool, she gets

$$xy - 150y = -190x$$

Selecting a y and dragging it left gives

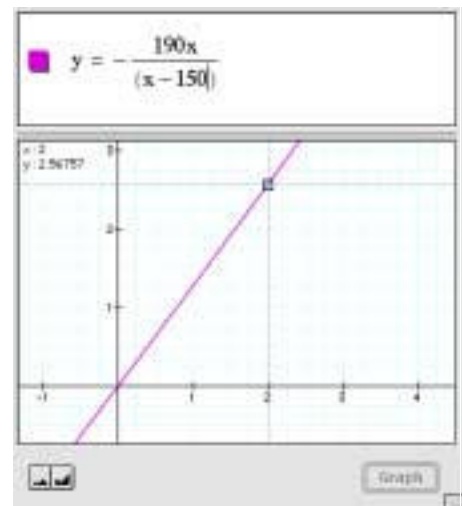
$$y(x - 150) = -190x$$

Whew! Now, with only one y she can isolate it:

$$y = -\frac{190x}{(x - 150)}$$

Behold! The resulting graph is in the margin.

Amanda is not surprised. It looks like a line, a simple proportion, just like the flagpole problem. She can use Graphing Calculator to get numerical predictions; when her tower is 2 cm tall, she expects the shadow to be about 2.6 cm long.



Of course, she will have towers taller than that. Her 2 cm blocks are easy to stack. She'll easily have towers 30 cm tall, so she zooms out as shown in the margin.

So with a 30-cm tower, she expects a shadow of about 47 cm.

But wait. The line no longer looks like a line. It looks like it's curving up. Now, concerned, she looks at the function she's plotting. It's got stuff in the denominator; it's not just a straight proportion like $y = kx$, or even

$$\text{Flagpole's height} = \frac{\text{Amanda's height}}{\text{Amanda's shadow}} \times \text{Flagpole's shadow}$$

She zooms out some more.

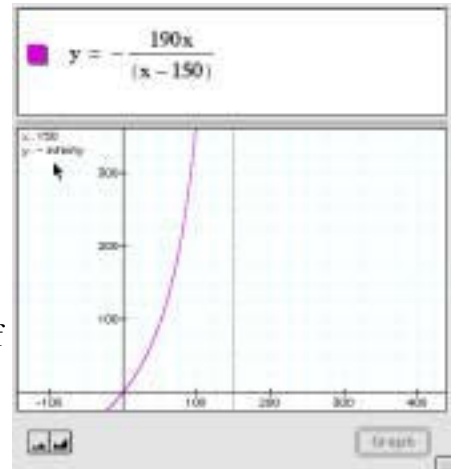
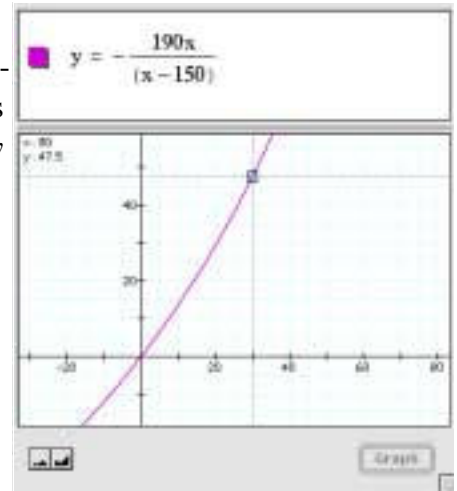
Not only does it curve up, it explodes—at $x = 150$. That makes sense *algebraically*, because that's where the denominator goes to zero. That's what usually makes functions blow up. But what does that mean in real life? It always helps to go back to the situation and make sense of it. Let's see, she thinks, $x = 150$ means a tower 150 cm tall. Why would the function blow up?

Then it hits her—that would be a tower as tall as the lamp. Of course the shadow would go on to infinity!

Amanda has discovered that the situation with the lamp casting a shadow is fundamentally different from the one with the sun. As she plans her physical experiment, she wonders if the data points she'll collect with towers to 30 cm tall will even show a curve, or if they will look like a straight line.

Things to do

- ❖ Find the line that's a good approximation to Salvador's function at $x = 100,000$ km. It's not $y = x$ (though it has the same slope). Explain its intercept.
- ❖ What if Amanda had chosen to move the tower of blocks instead of changing the height of the tower? Predict how the length of the shadow depends on the distance of the tower from the lamp.
- ❖ How is the situation with the lamp mathematically different from the one where the sun cast the flagpole's shadow?



Proportion Problems

If you've taken math sometime in the last century, this problem probably sounds familiar:

Which is a better deal, a 10-ounce box of cookies for \$1.99 or a 1-pound box for \$3.29?

This is a *proportion* problem. Graphing Calculator can help you solve problems like these. Let's do the cookies as an example.

You might rephrase "which is the better deal" as "which gives you more cookie *per dollar*." That's what math books mean by "a better deal." The "per" tells you that some proportion is going on. Let's let the left side be the 10-ounce package, and the right side the 1-pound package. You also have to know that there are 16 ounces in a pound.

Remember, we want cookies (or rather ounces) per dollar. So the ounces go on the top, and the price goes on the bottom.

❖ Type it in (without units):

$$\frac{10}{1.99} = \frac{16}{3.29} \quad \text{false}$$

Great! Graphing Calculator calculates the value of what's in the window *as long as there are no letters present*. So we know the ratios aren't equal.

❖ Now substitute > and < for the equals sign until you get true:

$$\frac{10}{1.99} > \frac{16}{3.29} \quad \text{true}$$

❖ The 10-ounce package is the better deal. How much better? Look at the ratios as you type them in:

$$\frac{10}{1.99} = 5.02513 \quad \text{versus} \quad \frac{16}{3.29} = 4.86322$$

Because of the way we set them up, those are the number of ounces you get per dollar. If we'd done it the other way,

$$\frac{1.99}{10} = 0.199 \quad \text{versus} \quad \frac{3.29}{16} = 0.205625$$

the results would be in dollars per ounce, that is, 19.9¢ per ounce for the 10-ounce box against 20.6¢ per ounce for the 1-pound box.

Suppose you wanted to sell a box of cookies that cost \$4.29. How many ounces of cookies should you put in it?

We have to make some assumptions. Since this is a math book, and we just had a cookie problem, we might assume that the price per ounce was the same as for the cookies in the last problem. It was about the same for the two.

- ◆ Let's set up a proportion using the 10-ounce box. The unknown—the weight of the new box—we leave as a letter, W .

$$\frac{1.99}{10} = \frac{4.29}{W}$$

- ◆ Now we have to solve for W . The easy way is to select the W , then choose **Isolate** from the **Math** menu.

$$\frac{1.99}{10} = \frac{4.29}{W} \text{ becomes } W = \frac{42.9}{1.99}$$

- ◆ Select and then choose **Calculate** from the **Math** menu.

$$W = \frac{42.9}{1.99} \text{ becomes } W = 21.5578$$

So there should be about 21.5 ounces in the box. If we do it with the 1-pound figures (\$3.29 and 16 ounces), we get 20.9 ounces in a box. What do you think? Should we give the customers a break with 21.5 ounces or continue the trend of worse-deal-the-bigger-the-box and give them only 20.5 ounces? You can decide, but you had better be able to show that you can set up the proportions to justify your decision.

Things to do

- ◆ More cookies. What about the jumbo 32-ounce size? How much should it cost? Of course it has to end in a 9.

- ◆ You count the cookies in a 10-ounce box. There are 23. How many cookies in a pound?
- ◆ How much does one cookie cost in the 10-ounce box?
- ◆ Belinda can drain her full 50-gallon bathtub in 8 minutes and 52 seconds. When her tub was only partly full one day, it took the water only five minutes to drain. How much water was in the tub? (You have to make an important assumption to do this problem. What is it? Do you think it's true? How could you tell?)
- ◆ Who reads faster, Beavis, who read the entire 875-page Mega-Comic edition of *Anna Karenina* in 4 hours and 12 minutes, or Dwayne, who stopped after 3 hours 35 minutes and got to page 726? (Dwayne missed the subway scene—too bad.)
- ◆ Is their reading speed realistic? Why or why not? (You might make some argument having to do with pages-per-minute, or minutes-per-page...)

Maximum-Minimum Problems

Problems that find the maximum or minimum of something are important in real life. If you use Graphing Calculator, you can transform the problem into a function, plot it, and see where it has a maximum or a minimum.

Here's a classic from practically every math textbook.

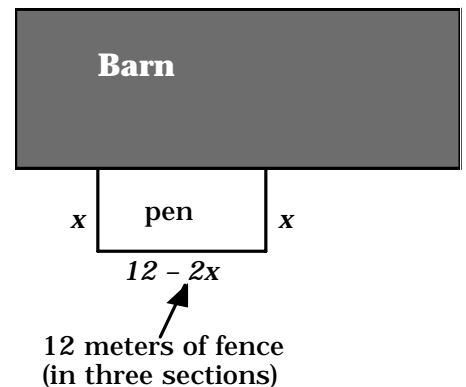
Ben is making a pen next to his barn for his pig, Agnes. He has 12 meters of fence. He can use the side of the barn for one side of the pen. What's the largest rectangular pen—the one with the biggest *area*—he can make with the 12 meters of fence he has?

It's always good to start with a picture.

Of course, we don't know the shape of the pen—that's what we're supposed to find out. But we do know that it's rectangular and that the length of the fence is 12 meters.

What will x and y be? We want to maximize the area. We'll make that y .

But what's x ? To find the area, we'll need the length and the width of the pen. We'll pick one to be x . Let's pick the width (in the picture). We'll make that x . Then the length of the pen is $12 - 2x$.



Note: This is something Graphing Calculator can't help you with. You have to understand the problem well enough to see that the length of the pen is $12 - 2x$.

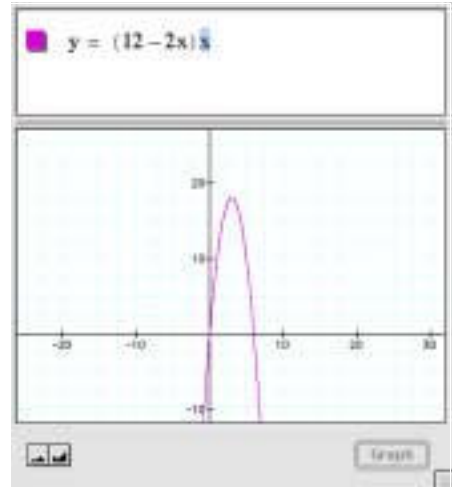
Now we're ready to write an equation for the area.

$$\text{Area} = \text{length} \times \text{width}$$

Or, in Graphing Calculator, given what we just said:

$$y = (12 - 2x)x$$

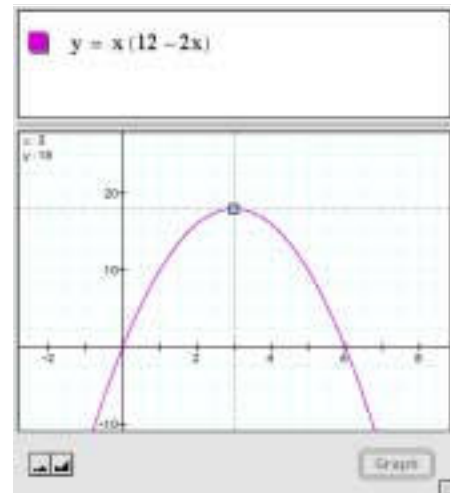
If we plot it, we get the screen in the margin.



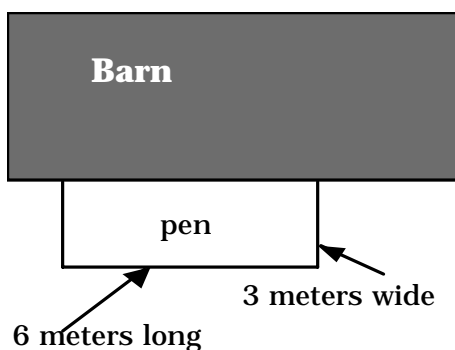
What does it mean? It's the area of the pen as it depends on the width. If the width of the pen is zero, it will be 12 meters long and have zero area (pretty stupid pen, Ben). If it is 6 meters wide, Ben won't have any fence left over for length, and it'll be zero again. Looks like the maximum is at 3 meters (which leaves 6 meters for length) with an area of 18 (we can read it off the graph or multiply 3 times 6).

Is it really the maximum? Click on the curve to get the curve tracker; move it up to the top. In the bottom picture in the margin, we've zoomed in.

We changed the horizontal scale of the graph without changing the vertical. (On the Macintosh, hold down the **option** key while you press the zoom buttons; on Windows, hold down **ctrl**.)



You'll notice that the tracker "sticks" to the maximum and makes a little clicking sound. You can read the x and y coordinates right off the screen. A scale drawing of the answer appears below.

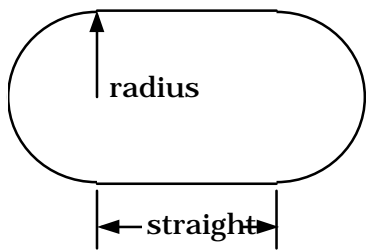


Things to do

- ◆ What if we'd chosen x for the length? Then there would be $(12 - x)$ meters of fence left over for the width, but there are two of those, so the width would've been $(12 - x)/2$. In that case, length times width would be $y = x\left(\frac{12 - x}{2}\right)$

Get that into Graphing Calculator and find the maximum. Show that it's really the same solution as the one we just got, even though you get different numbers.

- ◆ **No Barn, Ben.** Suppose Ben wants to make a freestanding rectangular pen with 12 meters of fence. He can't use the side of the barn. Now what shape makes the largest area? (It's a square. Can you prove it?)
- ◆ **The Oval Pen.** Same problem, except this time the pen has to be an oval, where we define an oval to be a rectangle with a semicircle attached to each end (like a running track).



The perimeter of this figure is fixed at 12 meters; find the dimensions of maximum area. One approach is to let x be the radius of the circle (and therefore half the width of the rectangle). Then you find the length of the rectangle by subtraction. Be sure to draw the figure.

- ◆ **The Capybara and the Sloth.** The Queen has heard about Ben's pen-making expertise, so she comes to him and asks him to make two pens: one for her capybara and one for her sloth. The capybara must have a circular pen, the sloth a square. The Queen is rich. She gives him 100 meters of fence to work with. She spoils her pets, so she is going to pave each pen with gold. But she is no dummy, so she instructs Ben to make the two pens in such a way that the sum of their *areas*—and therefore the cost of the gold paving stones—is a minimum.

Simultaneous Equations

You can use Graphing Calculator to solve simultaneous equations by substitution. It won't do your thinking for you, but it can eliminate copying mistakes.

When you have two simultaneous equations (equations P and Q) in two unknowns, one way to deal with them is to follow these steps:

- ❖ Solve equation P for one of the variables.
- ❖ Substitute that solution into equation Q.
- ❖ Solve equation Q for the remaining variable.
- ❖ Substitute that value into either P or Q to get the value of the first.

Here's an example. Mom says Bob and Denise have to share 117 jelly beans, but since Denise is older, she should get more. They agree that since Denise is 16 and Bob is 11, Bob should get 11/16 as many as Denise. (Bob is a sap, but never mind. This is a math problem...) The result is two equations:

$$B + D = 117 \text{ and } B = \frac{11}{16}D$$

Here is one way to use Graphing Calculator to solve simultaneous equations:

- ❖ Write down the two equations on paper (you won't get them both on the screen).
- ❖ Get the right-hand equation on the screen.
- ❖ Select the right-hand side.

$$B = \frac{11}{16}D$$

- ❖ Choose **Copy** from the **Edit** menu.

Now the expression for B is in the clipboard, ready to be pasted.

- ❖ Enter the other equation and select B .

$$B + D = 117$$

- ❖ Choose **Paste** from the **Edit** menu.

$$\frac{11}{16}D + D = 117$$

This substitutes the saved value for B into the equation. Now we have to solve this for D .

Note: If you select the first D and choose **Isolate Term**, you get

$$D = \frac{16(-D + 117)}{11}$$

Which may not help!

◆ Select and grab the first D .

$$\frac{11}{16}D + D = 117$$

◆ Drag it to the right until you get this:

$$\left(\frac{11}{16} + 1\right)D = 117$$

◆ Now choose **Isolate Term** with the D selected.

$$D = \frac{117}{\left(\frac{27}{16}\right)}$$

◆ **Select All** and **Calculate** to get an answer, $D = 69.3333$.

If Denise is nice, she'll take 69 and let Bob keep the rest. If you have been saving the equations in a word processor, you can retrieve them to find out how many Bob gets. Or you can just type in the subtraction:

$$117 - 69 = 48$$

So Bob gets 48.

Another Simultaneous Example

Here's one with just raw letters and numbers—no word problem. Find the values of x and y that satisfy:

$$2x + 3y = 10 \text{ and } 3x - 2y = 4$$

As in the previous section, pick an equation to solve (the first) and a variable (x). (It doesn't matter which, but you have to keep track.)

- ◆ Type in the equation, select x , and choose **Isolate Term**.

$$2x + 3y = 10 \text{ yields } x = \frac{-3y + 10}{2}$$

- ◆ Select and copy the right-hand side (choose **Copy** from the **Edit** menu).

- ◆ Now type in the second equation and select x .

$$3x - 2y = 4$$

- ◆ Paste the expression for x you copied above.

$$3 \frac{-3y + 10}{2} - 2y = 4$$

- ◆ Now the equation is in one variable. If you **Select All, Expand**, and so on, you get:

$$\frac{-9y + 30}{2} - 2y = 4$$

How will we combine the y s? You can mess around until it falls out, or you can think, "Get rid of the fractions and get the y s on one side."

- ◆ Grab a "2" on the left:

$$\frac{-9y + 30}{2} - 2y = 4$$

- ◆ Drag it to the right. Watch what happens on the way. You should end up with this:

$$(-9y + 30) - 4y = 2 \cdot 4$$

- ◆ Select the left and **Simplify** to get:

$$-13y + 30 = 2 \cdot 4$$

- ◆ Select y and **Isolate** it.

$$y = \frac{22}{13}$$

- ◆ Now, to find x , type in the first equation.

$$2x + 3y = 10$$

- ◆ Select y , and substitute (either by typing or pasting) the value we just got.

$$2x + 3\frac{22}{13} = 10$$

- ◆ **Isolate** x and you're done.

$$x = \frac{32}{13}$$

Things to do

Solve the following using Graphing Calculator. You may be able to get exact answers, or you may be able to make a graphs and see where it has some important property. Do something appropriate, but be sure to describe how you did it. In the first seven exercises, solve the simultaneous equations.

- ◆ Solve $3x + 5y = 0$ and $x - y = -2$.
- ◆ Solve $\frac{2x}{7} - 4y = 1$ and $5x - \frac{3y}{5} = 9$.
- ◆ Solve $x = \frac{8 - 9y}{3}$ and $x = \frac{5 - 6y}{2}$.
- ◆ Solve $y = x + 1$ and $y = \frac{1}{x}$.
- ◆ Solve $y = x^2 - 4x + 5$ and $y = x^2 + 2x - 1$.
- ◆ Solve $y = -x^2 - 4x - 1$ and $y = x^2 - 4x - 3$.
- ◆ Solve $y = \tan x$ and $y = \cos x$.
- ◆ Can you represent the smallest positive answer to the above question geometrically on the unit circle?

- ◆ Emmet, Francine, and Gus collect pogs. Emmet and Francine together have eighty-one pogs. Emmet and Gus have seventy-nine. Francine and Gus have one hundred two pogs all together. Guess the question, then answer it.

Equilibrium, Titration, and Logarithmic Axes

Graphing Calculator can even help in chemistry class. Practically any old calculator can do the bread and butter, but, as we shall see, Graphing Calculator can go a little further. This section also shows you how to get logarithmic axes for your plots.

Note: This section assumes you know—or remember—something about equilibrium calculations as they appear in high-school chemistry.

To refresh your memory, a *mole* of something (usually a molecule or ion) is 6.02×10^{23} of it. The *concentration* of something in a solution is the number of moles of it per liter of solvent (usually water). Concentrations are written with the thing inside square brackets, so that “[Na⁺]” means “the concentration of sodium ions in moles per liter.”

First, let’s look at silver chloride. Somebody drops a spoonful of silver chloride into a beaker containing 100 ml of distilled water at 25°C and stirs. What are the equilibrium concentrations of Ag⁺ and Cl[−]?

We look up the solubility product constant for this reaction:



That means that, at equilibrium:

$$[\text{Ag}^+][\text{Cl}^-] = 1.7 \times 10^{-10}$$

The two concentrations will be the same; call them c .

- ◆ In Graphing Calculator, enter $c^2 = 1.7 \times 10^{-10}$.
- ◆ **Isolate** c and we’re done.

Note: To enter the quantity in scientific notation, do not use the letter e as you would in many calculators or programs. In Graphing Calculator, $e = 2.71828\dots$, a constant! Instead:

❖ Type $1.7*10^{-10}$.

It's almost the same way you would say it.

Things to do

❖ Do the same calculation as in the example but use calcium carbonate, CaCO_3 , which is much cheaper than silver chloride. It's

$$K_{\text{sp}} = 3.9 \times 10^{-9} \text{ at } 25^\circ\text{C}.$$

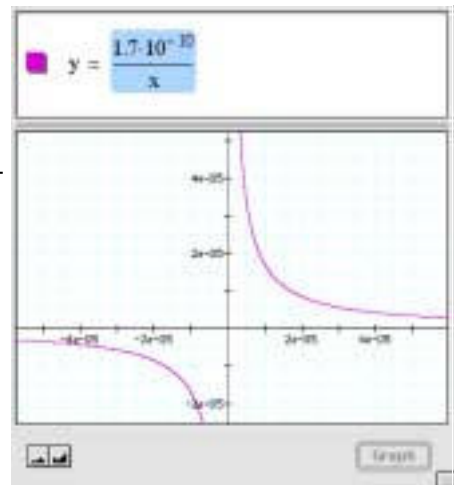
❖ Find a chemistry textbook and do practically any equilibrium calculation.

Using Functions and Logarithmic Axes

Suppose we take that beaker with the weak, saturated solution of silver chloride and add table salt, NaCl . Common sense (and Le Châtelier's principle) tells us that as we add chloride ions, some will combine with silver and precipitate out as AgCl , reducing the concentration of silver. How does the concentration of silver depend on the concentration of chloride? Let y be the silver concentration and x the chloride. Then:

$$[\text{Ag}^+][\text{Cl}^-] = xy = 1.7 \times 10^{-10}$$

We can isolate y and plot the result, which is unsatisfactory. The curve is so close to the x and y axes that we can't distinguish them without zooming a lot. You can see the result in the margin.



We need to do something to make the graph more meaningful. One tool we use for this is *logarithmic axes*. That is, instead of the silver concentration, we would rather get the logarithm of the silver concentration. That way, if the graph read “−6,” for example, the concentration would be 10^{-6} . And instead of the chlorine concentration, we'd rather look up its logarithm as well.

There are several ways to get such axes from where we are now. Here's one:

❖ Everywhere you see y , select it and substitute 10^y .

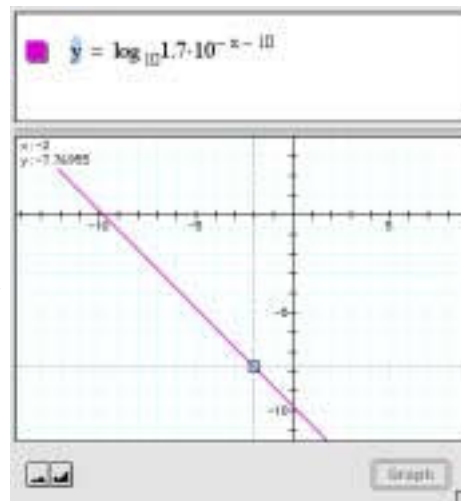
- ◆ Everywhere you see x , select it and substitute 10^x .

$$10^y = \frac{1.7 \cdot 10^{-10}}{10^x}$$

- ◆ Select the y and **Isolate** it.

When you graph the result (as in the margin), you get a log-log plot.

How do you read such a graph? For example, the selected point tells us that when the Cl^- concentration is 1.0×10^{-2} (its log is -2.0), the silver concentration will have gone down almost to 10^{-8} .



Things to do

- ◆ The coordinates in the preceding example are $(-2, -7.77)$ as you can see. Suppose “almost to 10^{-8} ” isn’t accurate enough? What is the actual concentration of Ag^+ to three places?
- ◆ Why did substituting 10^y and 10^x work for making logarithmic axes?
- ◆ The log-log graph looks like a straight line. Is it? Why or why not?

Titration: One More Example in Excruciating Detail

Chemistry texts often make you perform equilibrium calculations like those above, but in real life you don’t add salt to make the log of the concentration of the chloride ions -2.0 —you have to figure out the concentration. Some books deal with this when they discuss titration of acids and bases. And there is no acid-base pair closer to the hearts of chemistry students everywhere than those arch-enemies, hydrochloric acid (HCl) and sodium hydroxide (NaOH).

Here’s the setup: We’ve got 1 liter of 1 M NaOH (1 M = “one molar” = one mole per liter), and we’re going to add 1 M HCl . How does the concentration of the hydronium ion $[\text{H}_3\text{O}^+]$ change as we add the acid?

Here are some considerations:

- ◆ As we add solution, the total volume of solution will increase from 1 liter. If x is the amount we’ve added, the volume, V , is $(1 + x)$.

- ◆ As we add acid, we're adding H_3O^+ . Some will combine with the OH^- to make water. They will combine in a way that keeps

$$[\text{H}_3\text{O}^+][\text{OH}^-] = 1.00 \times 10^{-14}$$
- ◆ At the beginning, therefore, since $[\text{OH}^-] = 1$ (it's a 1-molar solution), $[\text{H}_3\text{O}^+] = 1.00 \times 10^{-14}$.
- ◆ If we add 1 liter of acid to the 1 liter of base solution, the result had better be neutral (but salty). We expect that when x is 1, $[\text{H}_3\text{O}^+]$ should be 1.00×10^{-7} .
- ◆ Because we're working only with 1-molar solutions, we know that there's 1 mole of OH^- initially, and there are x moles of H_3O^+ added at any time (though some will be recombined into water).

Now we will set up four equations we need to solve in order to create the function (hydronium concentration as a function of acid added). We omit the grunt work so you can do it yourself with Graphing Calculator.

Let y be the concentration of hydronium in moles per liter, $[\text{H}_3\text{O}^+]$, and x be the amount of 1 M HCl added in liters. We will need some intermediate variables: V will be the total volume of solution and h will be $[\text{OH}^-]$. Finally, let w be the number of moles of hydronium (or OH^-) that have recombined into water. Then the following equations are true:

$$yh = 1.00 \times 10^{-14} = K_w \quad (\text{equilibrium})$$

$$V = 1 + x \quad (\text{conserving volume})$$

$$x = yV + w$$

This is a hard one. We add x liters (and therefore x moles) of hydronium. Some of it recombines into water. Some is in solution, in equilibrium with the hydroxyl. yV —the hydronium concentration times the volume—is the total number of moles of hydronium that are still in solution. And w is the number that have recombined. They add up to the total amount x .

$$1 = hV + w$$

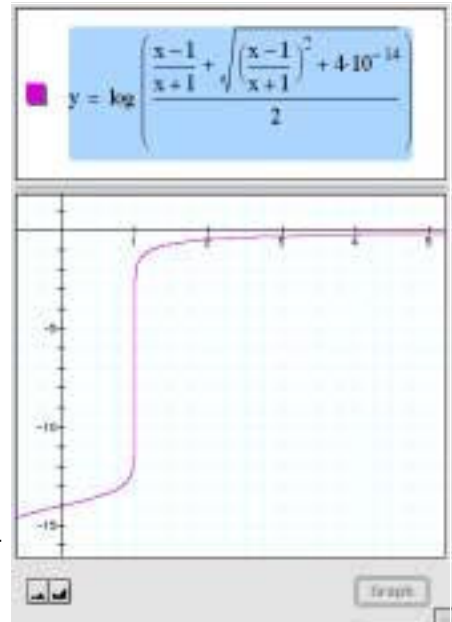
Same reasoning for hydroxyl concentration, $[\text{OH}^-]$. The number of moles that recombine, w , is the same as in the previous equation, since one OH^- hooks up with one H_3O^+ .

We combine these equations to get y as a function of x . Graphing Calculator will help you except for the part where you apply the quadratic formula. Here's what we got (we took the positive root):

$$y = \frac{\frac{x-1}{x+1} + \sqrt{\left(\frac{x-1}{x+1}\right)^2 + 4 \times (1.00 \times 10^{-14})}}{2}$$

To enter this into Graphing Calculator, you may want to change your preferences to make the font size smaller.

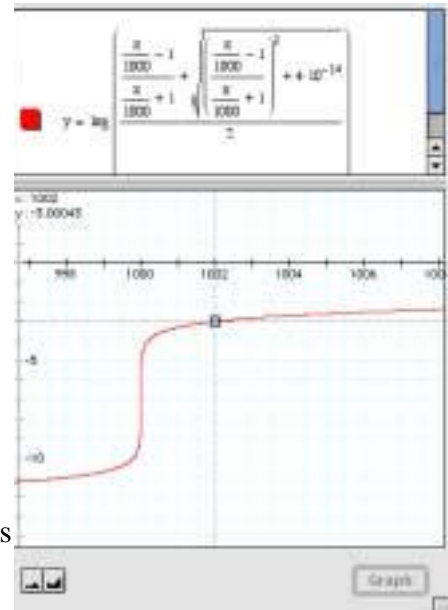
The first graph you get is just as unsatisfactory as the one of the silver chloride; this time we'll use a log plot only in the y-direction, as in the illustration in the margin.



Things to do

Since, in the graph, y is the log of $[\text{H}_3\text{O}^+]$ and pH is defined to be the negative of that, we can read pH right off the graph. Remember, x is the number of ml of HCl solution added.

- ❖ Type in the equation as written in the upper screen in the margin.
- ❖ Verify that the pH is correct at the beginning ($x = 0$, before you've added any NaOH) and neutral ($\text{pH} = 7$) when you've added 1 liter.
- ❖ What is the pH when you've added 2 liters of HCl solution? (We get about 0.5—very acidic!) Is that correct? Verify it by hand.
- ❖ So we can read x in milliliters instead of liters, divide every occurrence of x by 1000 (apply a horizontal stretch factor) as shown in the margin.
- ❖ Zoom to the transition area as shown. Verify that the solution is neutral when you have added 1000 ml of solution.



- ◆ The picture shows that the pH is 3 (also very acidic) when you've added 1002 ml. Can that be right? Explain.
- ◆ How many milliliters of hydrochloric acid separate a pH of 10 from a pH of 4? A pH of 8 from one of 6?
- ◆ Use algebra to solve the four equations on page 139 and find the function we have been graphing. Use substitution to eliminate the variables you don't want—the select-and-type-over method works. You may need the quadratic formula.

Note: When Graphing Calculator takes the log of a tiny concentration, it can become unsure whether it's greater than zero. If your graphs are discontinuous and give you “not a number” for y —use simple graphing. (On the Macintosh, press **⌘-option-s**. On Windows, press **ctrl-shift-S**). To reverse this, use best graphing. (On the Macintosh, press **⌘-option-b**. On Windows, press **ctrl-shift-B**).

Orbital Mechanics

You can use Graphing Calculator's graphs and numerical approximations to help with physics. Let's use algebra to derive one of Kepler's laws—but we'll cheat and use Newton's work.

In a circular orbit, gravity is balanced by centrifugal force. Centripetal acceleration of an object moving in a circle is

$$a_c = \frac{v^2}{r}$$

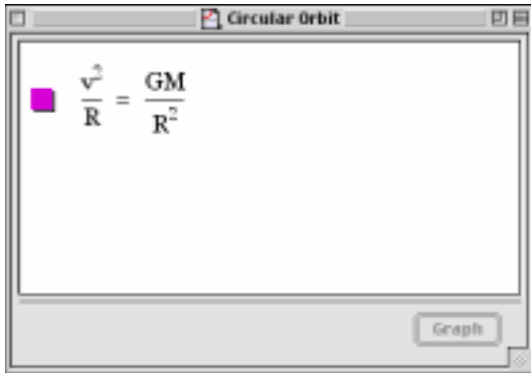
where v is the speed of the object and r is the radius of the circle it's moving in. The acceleration due to gravity is

$$a_g = \frac{F}{m} = \frac{GM}{r^2}$$

where G is the gravitational constant, r is again the radius of the orbit, and M is the mass of the central body, in this case, the sun.

We want to find a relationship between the radius of a planet's orbit and the length of its year—the amount of time it takes to make one circuit around the sun.

We begin by setting these two opposing accelerations equal to one another since the planets are in circular orbits. You can see that below:



We'll show you the whole process so you can see it.

How did we choose to do what we did? What did we need to know? We could've just messed around with things and maybe stumbled into it, but in this case, knowing that we wanted the relationship between period and radius, we knew that we had to get period *into* the equation, and we had to get velocity *out* of it.

That's easily done by realizing that a year is just one planetary trip around the sun, and that the circumference of the orbit is the distance traveled. Then we had to remember that

$$\text{distance} = \text{velocity} \times \text{time}$$

relates distance to velocity and time.

Ultimately, as you'll see, we got Kepler's result, $T^2 \propto r^3$

To do what we did

- ◆ Starting with the equation on page 141, select v and **Isolate**.

$$v = \frac{\sqrt{G} \cdot \sqrt{M}}{\sqrt{r}}$$

- ◆ We know that $v = C/T$, where C is the circumference of the orbit, and T is the period—the length of the year. Make the substitution.

$$\frac{C}{T} = \frac{\sqrt{G} \cdot \sqrt{M}}{\sqrt{r}}$$

- ◆ That circumference is $2\pi r$, so select the C and replace it.

$$\frac{2\pi r}{T} = \frac{\sqrt{G} \cdot \sqrt{M}}{\sqrt{r}}$$

- ◆ Select T and **Isolate**. Almost there!

$$T = \frac{2\pi r^{3/2}}{\sqrt{G} \cdot \sqrt{M}}$$

- ◆ **Select All** and square it by typing $\wedge 2$. You get the unorthodox (but clear) “square both sides” notation.

$$T = \frac{2\pi r^{3/2}}{\sqrt{G} \cdot \sqrt{M}} \text{ becomes } \left(T = \frac{2\pi r^{3/2}}{\sqrt{G} \cdot \sqrt{M}} \right)^2$$

- ◆ **Select All** and **Simplify**. This is it:

$$T^2 = \frac{4\pi^2 r^3}{GM}$$

Things to do

- ◆ The acceleration of gravity at the surface of the earth is 981 cm/sec^2 . The earth’s circumference is $40,000 \text{ km}$. What’s GM , the mass of the earth times the gravitational constant?
- ◆ Many communications satellites are geosynchronous—they have orbital periods of 24 hours, so they stay over the same place on the globe all the time. What’s the radius of their orbits?
- ◆ Suppose you were out in space and a paper clip were orbiting you at a distance of 10 meters. How long would one orbit take? (You may need additional information. See your physics book.)

Solutions

I hate giving answers. With a program like Graphing Calculator, you know when you're right. And a lot of times, if we see an answer, we stop thinking. But sometimes it helps to have a little external validation, huh? So here are some solutions to the exercises in this book.

From “Putting transformations together” on page 38

The four parabolas are $y = (x-1)^2 - 1$, $y = \frac{1}{4}(x-1)^2 - 1$

$y = -2(x+2)^2 + 2$, and $y = 5(x-0.1)^2$.

From “Once more, with waves” on page 40

The four waves are $y = \sin(\pi x) + 1$, $y = \sin\left(\frac{x}{10}\right)$, $y = \frac{\pi}{3}\sin\left(x + \frac{3}{2}\right)$

$y = 2\sin(x-1)$.

From “The difference between horizontal and vertical”, question on page 44

The question about the parabola is tricky. The coefficient has to be *exactly* 2 to make a parabola.

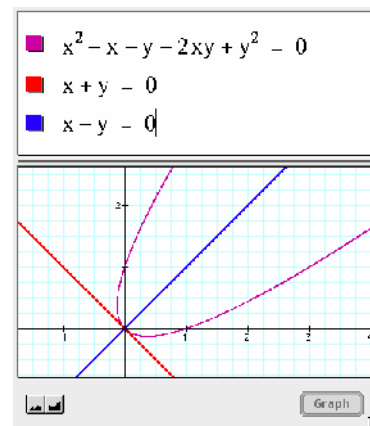
As to why it's a parabola, consider this manipulation:

$$\begin{aligned}x^2 - x - y - 2xy + y^2 &= 0 \\x^2 - 2xy + y^2 &= (x + y) \\(x - y)^2 &= (x + y)\end{aligned}$$

So if we defined two new variables

$$\begin{aligned}u &= x + y \\v &= x - y\end{aligned}$$

We would have the equation $u = v^2$, which is at least a quadratic in v . But is it a parabola in x - y space? Yes. In the illustration, you can see the axes of this new coordinate system. These might help you imagine all lines of constant u or constant v , which are perpendicular to one another. You could make a grid out of them. It would be just like the regular Cartesian grid except that it's tilted 45° and *the grid spacing is $\sqrt{2}$ larger*.



From “Multiplying functions: envelopes” on page 50

- ◆ The wave with the decreasing envelope is $y = \sin(5\pi x)2^{-x}$.
- ◆ As to making a moving wave packet, there are lots of ways to do it. Consider the envelope. Another “humpy” function is $y = e^{-x^2}$, which is the “bell” shape of the normal distribution. The key is to make n control the horizontal position of the envelope (whether you also transform the underlying wave is up to you). So we need an envelope like

$$y = e^{-(x-n)^2}$$

(using the subtract-the-horizontal-offset-from-every-occurrence-of- x principle we developed in “Translations, the minus sign, and symmetry” on page 41).

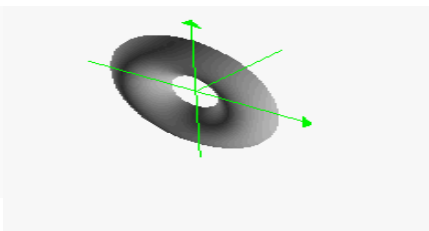
Just multiply that by a wave—any wave as long as it’s fast enough—and you’re done!

- ◆ Now, how do you make the beats picture—which is a sum—into an envelope situation—which is a product? I won’t spoil your fun, at least not completely. here is part of the answer:
 - The frequency of the envelope is *half the difference between the two frequencies of the waves*. You can prove this to yourself by experimenting with n .
 - The frequency of the underlying wave is the *average* between those two frequencies.

From “Surfaces of revolution” on page 67



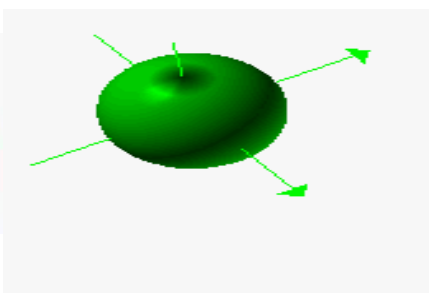
$$r = \cos((2\pi z) - 1)$$



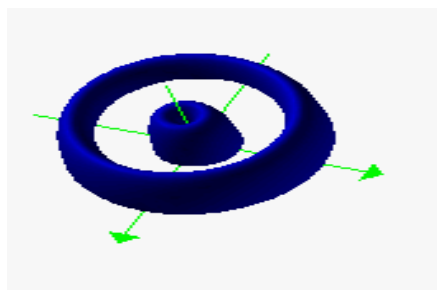
$$z = (r - 1)^2 - \frac{1}{3}, z < 0$$



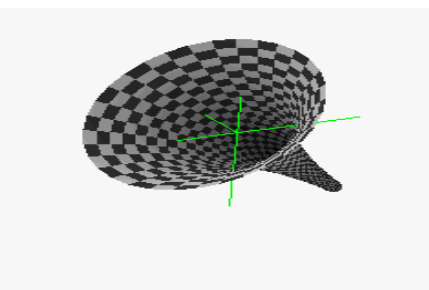
$$r = \frac{z}{2} + \frac{3}{2}$$



$$r = \sqrt{1 - z^2}$$



$$z = \sqrt{\sin(2\pi r)}$$

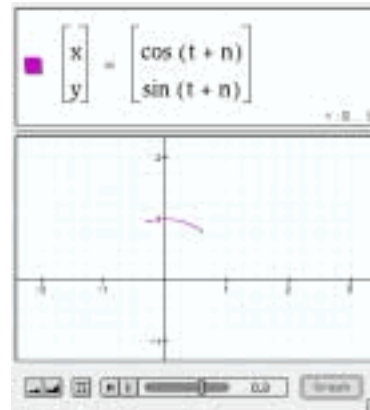


$$r = 2^z$$

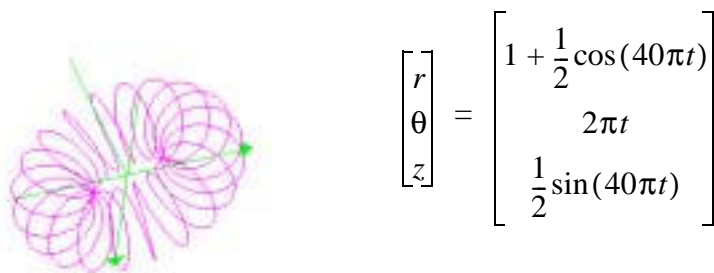
From “Making parametric curves in Graphing Calculator” on page 79

- Add to the x -component, as in $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos((2\pi t) + 1) \\ \sin(2\pi t) \end{bmatrix}$

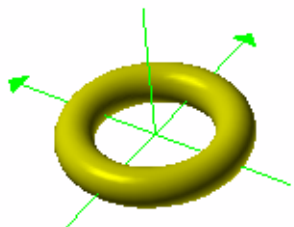
- Get a vertical stretch by multiplying the y -component by some number, for example, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(2\pi t) \\ 2 \sin(2\pi t) \end{bmatrix}$
- “Translating” the parameter t slides the range of the parameter. If you’re looking at a complete circle, you won’t see any difference. But if you’re looking at *part* of a circle, translating t will get you a different part, as in the illustration at right.
- “Stretching” the parameter t extends the range of the parameter. When we multiplied t by $2\frac{1}{4}$ everywhere in the equation to complete the circle, we were doing a stretch by that same factor.



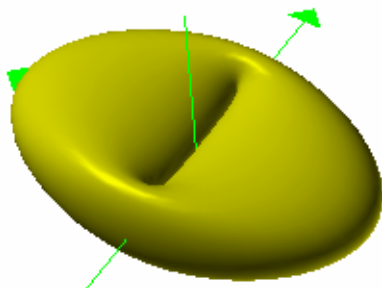
From “Making parametric curves in Graphing Calculator” on page 79



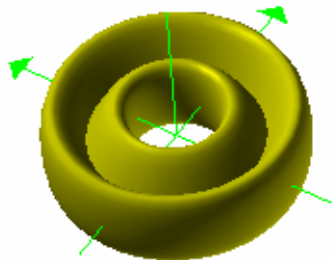
The Modified Donut questions from page 88



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos 2\pi u \left(1 + \frac{\cos(2\pi v)}{4} \right) \\ \sin 2\pi u \left(1 + \frac{\cos(2\pi v)}{4} \right) \\ \frac{\sin(2\pi v)}{4} \end{bmatrix}$$

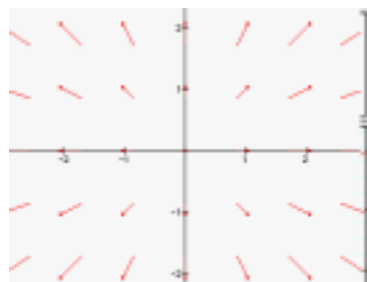


$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos 2\pi u \left(1 + \frac{\cos(2\pi v)}{4} \right) \\ \sin 2\pi u (1 + \cos(2\pi v)) \\ \frac{\sin(2\pi v)}{2} \end{bmatrix}$$

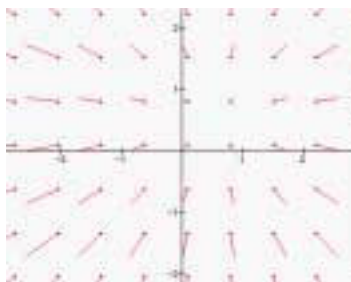


$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos 2\pi u \left(1 + \frac{\cos(2\pi v)}{2} \right) \\ \sin 2\pi u \left(1 + \frac{\cos(2\pi v)}{2} \right) \\ \frac{\sin(4\pi v)}{2} \end{bmatrix}$$

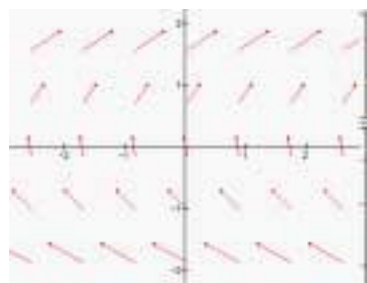
From “Vector fields” on page 106



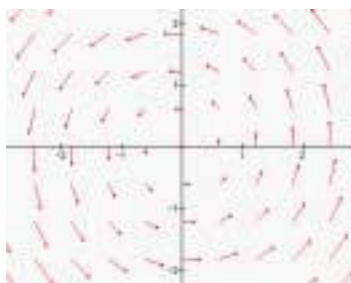
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -(x - \frac{1}{2}) \\ -(y - \frac{1}{2}) \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

Numerics

- 2-Vector command 79
- 3D Background
 - changing 17
- 3D parametric curves 82
- 3D surfaces 77
 - slicing 70
- 3-Vector command 82, 92

A

- absolute value 10
- absolute value marks 104
- adding functions 45, 46
- algebra
 - preserving equality 43
 - rules of 116
 - word problems 107
- animation slider 20
- area
 - equation for 130
 - for circle 131
 - for oval 131
- average speed
 - figuring 113
- axes
 - letters for 22
 - scaling 3
 - telling which is which 61
 - zooming independently 3, 123

B

- bases
 - of logarithms 10
- beats 46
- best graphing 141
- Boolean expressions 93

C

- calculus 99
- Cartesian coordinate system
 - 3D graphs in 62
- ceiling function (ceil) 19
- circles 29
- circumference 108
- clipping documents
 - creating on Macintosh 4
- colors
 - and resolution 92
 - for graph information 93
 - for graphs 22
 - for surfaces 91
 - for vectors 92
 - using HSV 106
- combination functions 99
- composition
 - of functions 99
- concentration 136
- conditions 93
- conic sections 31, 74
- conserving volume
 - equation 139

- constants
 - changing letters for 23
 - defining 23
- contour plots 106
- converting units 110
- coordinate systems
 - graphing in 62
- coordinates
 - editing range 63
- Copy As Text command 4
- Copy command 4
- Copy Graph command 4
- cylindrical coordinates 65, 78

D

- defining parameters 22, 23
- density plots 106
- dependent variables 125
- Derivative command 100, 102
- difference functions 47
- differential equations 102
- dilations 35
 - squishes 37
 - stretches 37
- dimensional analysis 109
- direct substitution 112
- distance
 - equation for 142
- division 9
- drag algebra 43
 - preserving equality 43
- Draw Graph command 7

E

- envelopes
 - multiplying functions 50
- equations
 - copying and pasting 4
 - two on the same curve 79
- equilibrium 136
- Expand command 101
- exponential functions 19, 44
- exponents 11
- expressions
 - copying and pasting 101

F

- factorial 10
- floor function 19
- formulas
 - entering 9
- Fourier series 48
- fractions 11
 - adding 117
- function arguments
 - entering 98
- functions
 - adding 45, 46
 - and transformations 35
 - combinations 99
 - composing 99
 - defining your own 98
 - multiplying 50
 - tracing 22

G

- geometric series 96
- Graph button 7
- graph pane 3
- graphing
 - more than one function 22
- Graphing Calculator
 - as calculator 12
 - entering formulas 9
 - reader 1
 - selection 7
 - versions 1
 - window 3
- graphs
 - changing colors for 22
- Greek letters
 - entering 19

H

- helices 83, 88
- Help command 4
- horizontal translation 41
- horizontal translations
 - for parametric equations 80
- hyperbola
 - 3D 76
- hyperbolic functions 19

I

- if statements 93
- implicit functions 75
- independent variables 124
- inequalities
 - and functions 28
 - and transformations 28
 - graphing 26
 - using for circles 25
- insertion point (cursor) 7
- integer function (See floor) 19
- integral server 103
- integrals 103
- integration 103
- interpreting results 108
- intersections
 - between surfaces 74
 - tracing in graphs 22
- Inverse
 - on keypad 10
- inverse functions 19
 - entering 19
- inversions 35
- Isolate Term command 17, 119

K

Kepler's laws 141
keypad 10

L

Lissajous figures 81
logarithmic axes 136, 137
logarithmic functions 19
logarithms 10

M

math pane 3
maximum-minimum 129
mole 136
multiplying by 1 110
multiplying functions 50

N

New command 10
New Math Expression command 22

O

ODEs 102
operating
 on whole equations 118
operators 8
orbital mechanics 141
ordinary differential equations 102

P

parabolas 25
 changing with sine 46
 entering functions for 43
 tilted 44
paraboloid 63
parameterizations 41
parameters
 changing letters for 23
 defining 23, 39
 different values for 20
 editing 39
 editing range 63
 represented on surfaces 63
parametric curves 79
 letter for 23
parametric equations 79
 without trig functions 82
parametric surfaces
 letters for 23
parentheses 9, 10
 and function arguments 98
partial derivative 100
periodic waves
 transforming 40
pi 19
planes
 in space 76
polar coordinates 24
 letters for 22
polynomials
 graphing 27
 shaded areas 27

Preferences command 17
problems
 maximum-minimum 129
proportions
 setting up 107

Q

quantity
 in scientific notation 137

R

radians
 and trig functions 19
radius of the earth 121
range
 changing 26
reciprocals 35
relativistic factor
 equation for 118
Reset Graph command 4
resolution
 and color 92
 in 3D graphs 62
results
 interpreting 108
RGB values
 for surfaces 91
rotating
 around specific axes 61

S

Save command 4
Select All command 7
selection 7
series of functions
 adding 95
Show Keypad command 10
sigma 95
significant figures
 changing number of 17
signum 19, 103
similar triangles 124
simple graphing 141
sine
 changing with parabolas 46
sinusoids
 adding 46
slider values 21
Slider Values command 23
special relativity 118
spherical coordinates 78
 letter for 23
spherical-polar coordinates
 3D graphs in 62
square waves 48
substitution 113
Summation command 50, 95
summations 95, 96
 entering 50
surfaces
 colors for 91

- surfaces of revolution 66, 67
- surfaces parallel to the x-axis
 - graphing 70
- surfaces parallel to the y-axis
 - graphing 71
- symbol manipulation
 - to change equations 107
- symbolic calculus 99
- symbols
 - and problems 107
- symmetry 41

T

- terms
 - isolating 17
- theta 26
- titration 136, 138
- tori 84, 85, 86, 87, 88
- total and partial derivatives 101
- tracer
 - coordinates 3
 - on graphs 3
 - zooming to 3
- Tracing functions 22
- Transform command 18
- transformations 34, 38, 40
 - combining 38, 40
 - dilations 37
 - types 35
- translations 35, 41
- transparency
 - for surfaces in graphs 62

- trigonometric functions 18, 19
 - transforming 40
- trigonometric inequalities 28

U

- units
 - abbreviating 111
 - converting 110

V

- variables 22
- vector fields 104
- vectors 79, 84
- vertical translation 41

W

- waves 40
- word problems 107

Z

- zoom buttons 3
- zooming 3
 - axes 3
 - to tracer 3